Quiz 9

Directions: Complete as many problems as you can during section. This is a low-stakes "quiz" - you will not be penalized if you do not finish every problem.[1] You may collaborate with other members of the class, consult the TA, and consult the textbook or lecture notes. Scan and save your work as a `.pdf` file and submit via https://canvas.ucsc.edu. You have until midnight to scan and submit your work.

This quiz will give you some practice identifying poles and computing their residues. For reference, here are some theorems we saw in the lecture.

Theorem 1. An isolated singularity $z_0 \in \mathbb{C}$ of a function f is a pole of order m if and only if there exists a function $\varphi(z)$ that is analytic and nonzero at z_0 such that

$$f(z) = \frac{\varphi(z)}{(z - z_0)^m}.$$

In this case, the residue can be computed via

$$\operatorname{Res}_{z=z_0} f(z) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!}.$$

Theorem 2. Suppose that functions $p(z), q(z)$ are analytic at z_0, $p(z_0) \neq 0$, and $q(z)$ has a zero of order m at z_0. Then the quotient $\frac{p(z)}{q(z)}$ has a pole of order m at z_0. In this case, it may be convenient to compute the residue using the formula from PSet 8 P1.

Theorem 3. Suppose that functions $p(z), q(z)$ are analytic at z_0, $p(z_0) \neq 0$, $q(z_0) = 0$, and $q'(z_0) \neq 0$. Then the quotient $\frac{p(z)}{q(z)}$ has a simple pole at z_0 and the residue is given by

$$\operatorname{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$

Q1. Use Theorem 1 to show that function $f(z) = \left(\frac{e^z}{2z+1}\right)^3$ has a pole of order $m = 3$ at $z = -\frac{1}{2}$ and compute the residue.

Solution. Taking $\varphi(z) = \frac{e^{3z}}{8}$, we see that

$$f(z) = \frac{e^{3z}}{8(z + \frac{1}{2})^3} = \frac{\varphi(z)}{(z + \frac{1}{2})^3}.$$

Moreover, $\varphi(z)$ is clearly entire and nonzero at $z = -\frac{1}{2}$. According to Theorem 1, $z = -\frac{1}{2}$ is a pole of order $m = 3$ and the residue is given y

$$\operatorname{Res}_{z=-\frac{1}{2}} f(z) = \frac{\varphi^{(2)}(-1/2)}{2} = \frac{9}{16} e^{-\frac{3}{2}}.$$

☕

Q2. Use Theorem 2 to show that the function $f(z) = \frac{1}{z \sin z}$ has a pole of order $m = 2$ at $z = 0$. Use the formula from PSet 8 P1 to compute the residue. You can use L'Hospitals rule to compute the limit.

[1] However, it is **essential** that you understand how to solve each problem, so I **highly recommend** that you continue to work on any unfinished problems throughout the week. You can also post questions on the "quizzes/section" stream on Zulip: https://math103a.zulipchat.com.

Solution. Set $p(z) = 1$ and $q(z) = z \sin z$. Evidently, both $p(z)$ and $q(z)$ are entire and $p(0) \neq 0$. Moreover, $q(z)$ has a zero of order $m = 2$ at $z = 0$. To see this, just observe that

$$q(0) = 0, \quad q'(0) = (\sin z + z\cos z)|_{z=0} = 0, \quad \text{and} \quad q''(0) = (2\cos z - z\sin z)|_{z=0} = 2 \neq 0.$$

According to Theorem 2, we conclude that $f(z)$ has a pole of order $m = 2$ at $z = 0$. To compute the residue, we use the formula from PSet 7. We have

$$\begin{aligned}
\operatorname{Res}_{z=0} f(z) &= \lim_{z\to 0} \frac{d}{dz}\frac{z^2}{z\sin z} \\
&= \lim_{z\to 0} \frac{d}{dz}\frac{z}{\sin z} \\
&= \lim_{z\to 0} \frac{\sin z - z\cos z}{\sin^2 z} \\
&\overset{L'H}{=} \lim_{z\to 0} \frac{z\sin z}{2\sin z\cos z} \\
&= \lim_{z\to 0} \frac{z}{2\cos z} \\
&= 0.
\end{aligned}$$

☕

Q3. Use Theorem 3 to show that $f(z) = \tan z$ has a simple pole at each point $z_k = \pi/2 + k\pi$, $k \in \mathbb{Z}$ and the compute the residue.

Solution. Let $p(z) = \sin z$ and $q(z) = \cos z$. Then $p(z_k) = (-1)^k \neq 0$, $q(z_k) = 0$, and $q'(z_k) = -\sin z_k = (-1)^{k+1}$. By Theorem 3, z_k is a simple pole the residue is given by

$$\operatorname{Res}_{z=z_k} \tan z = \frac{p(z_k)}{q'(z_k)} = -1.$$

☕