# Quiz 8

**Directions:** Complete as many problems as you can during section. This is a low-stakes "quiz" - you will not be penalized if you do not finish every problem.[1] You may collaborate with other members of the class, consult the TA, and consult the textbook or lecture notes. Scan and save your work as a `.pdf` file and submit via https://canvas.ucsc.edu. You have until midnight to scan and submit your work.

**Q1.** Let $C$ be the positively oriented circle $|z| = 3$. Use residue theory to compute $\int_C f(z)\,dz$ for each function $f(z)$.

(a) $f(z) = \dfrac{e^{-z}}{z^2}$;

(b) $f(z) = \dfrac{e^{-z}}{(z-1)^2}$;

(c) $f(z) = \dfrac{z - \sin z}{z}$.

*Solution.* For illustration of results from the most recent lecture, I will compute the residue in two ways (where applicable): first using the Laurent series, and second using the characterization of poles.

(a) The function has a single isolated singularity at $z = 0$, which lies interior to $C$. Hence, by Laurent's theorem (or the residue theorem) $\int f(z)\,dz = 2\pi i \operatorname{Res}_{z=0} f(z)$. For illustration, we compute the residue in two ways.

First, we compute the Laurent series of $f(z)$ on the annulus $0 < |z| < \infty$. Since $e^z$ is entire, the MacLaurin series for $e^z$ holds for all $z \in \mathbb{C}$, in particular, for $-z$. Hence, on $0 < |z| < \infty$, we can write

$$\frac{e^{-z}}{z^2} = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n z^{n-2}}{n!}.$$

The residue is the coefficient of the term occuring when $n-2 = -1$. Hence, $\operatorname{Res}_{z=0} f(z) = -1$.

Alternatively, you can write $\varphi(z) = e^{-z}$ and realize that $\varphi$ is analytic and nonzero at $z = 0$. Since $f(z) = \frac{\varphi(z)}{z^2}$, $z = 0$ is a pole of order $m = 2$ and hence the residue is given via

$$\operatorname{Res}_{z=0} f(z) = \frac{\varphi'(0)}{1!} = -1.$$

We conclude that $\int_C f(z)\,dz = -2\pi i$.

(b) The function has a single isolated singularity at $z = 1$, which lies interior to $C$. Hence, by Laurent's theorem (or the residue theorem) $\int f(z)\,dz = 2\pi i \operatorname{Res}_{z=1} f(z)$. Again, we compute the residue in two ways.

First, we compute the Laurent series of $f(z)$ on the annulus $0 < |z-1| < \infty$. Since $e^z$ is entire, the MacLaurin series for $e^z$ holds for all $z \in \mathbb{C}$, in particular, for $z - 1$. Hence, on $0 < |z| < \infty$, we can write

$$\frac{e^{-z}}{(z-1)^2} = \frac{e^{-1}e^{-(z-1)}}{(z-1)^2} = \frac{e^{-1}}{(z-1)^2}\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{n!} = e\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^{n-2}}{n!}.$$

[1] However, it is **essential** that you understand how to solve each problem, so I **highly recommend** that you continue to work on any unfinished problems throughout the week. You can also post questions on the "quizzes/section" stream on Zulip: https://math103a.zulipchat.com.

Clearly, $\operatorname{Res}_{z=1} f(z) = -e^{-1}$.

Alternatively, you can write $\varphi(z) = e^{-z}$ again. The same argument as above reveals that $z = 1$ is a pole of order $m = 2$ and hence

$$\operatorname{Res}_{z=1} f(z) = \frac{\varphi'(1)}{1!} = e^{-1}.$$

We conclude that $\int_C f(z)\, dz = \frac{2\pi i}{e}$.

(c) The function has a single isolated singularity at $z = 1$, which lies interior to $C$. Hence, by Laurent's theorem (or the residue theorem) $\int f(z)\, dz = 2\pi i \operatorname{Res}_{z=0} f(z)$. This time, we cannot compute the residue using the characterization of poles because, as we will see, the singularity is removable.

First, we compute the Laurent series of $f(z)$ on the annulus $0 < |z| < \infty$. Since $\sin z$ is entire, we can use the MacLaurin series which is valid for all $z \in \mathbb{C}$. On $0 < |z| < \infty$, we have

$$\frac{1}{z}\left(z - \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}\right) = -\sum_{n=1}^{\infty}(-1)^n \frac{z^{2n}}{(2n+1)!}.$$

Evidently, $\operatorname{Res}_{z=0} f(z) = 0$.

Now, notice that the principal part of $f$ at $z = 0$ is zero. By definition, this means that $f$ has an isolated singularity at $z = 0$, not a pole. It would be a mistake to write $\varphi(z) = z - \sin z$ and try to apply the characterization used in the preceding two problems. The problem is, that although $\varphi$ is analytic at $z = 0$, we have $\varphi(0) = 0$ - so the theorem does not apply.

We conclude that $\int_C f(z)\, dz = 0$.