Quiz 7

Directions: Complete as many problems as you can during section. This is a low-stakes "quiz" - you will not be penalized if you do not finish every problem.[1] You may collaborate with other members of the class, consult the TA, and consult the textbook or lecture notes. Scan and save your work as a `.pdf` file and submit via https://canvas.ucsc.edu. You have until midnight to scan and submit your work.

Q1. For a complex number $z_0 \in \mathbb{C}$, define the annular domain

$$A(z_0, R_1, R_2) = \{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}.$$

Compute the Laurent series for $f(z) = \frac{1}{(z-1)(z-2)}$ on each annulus:

(a) $A(0, 0, 1)$;
(b) $A(0, 1, 2)$;
(c) $A(0, 2, \infty)$;
(d) $A(1, 0, 1)$;
(e) $A(2, 0, 1)$.

In each case, write the Laurent series in the form (some terms may be zero):

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z - z_0)^n}, \qquad (R_1 < |z - z_0| < R_2).$$

Solution. First, use the partial fraction decomposition:

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}.$$

(a) Since $f(z)$ is analytic at 0, f is analytic on $|z| < 1$ and hence there is a MacLauren series on that disk. In this case, this coincides with the Laurent series. We have

$$\begin{aligned} f(z) &= -\frac{1}{2}\frac{1}{1 - \frac{z}{2}} + \frac{1}{1 - z} \\ &= -\frac{1}{2}\sum_{n=0}^{\infty} \frac{z^n}{2^n} + \sum_{n=0}^{\infty} z^n && (|z| < 1) \\ &= \sum_{n=0}^{\infty} (1 - 2^{-n-1}) z^n. \end{aligned}$$

(b) Since f is analytic on the annulus $A(0, 1, 2)$, it has a Laurent series there. The annulus is centered at 0 - so we look for a Laurent series in powers of z. The condition $1 < |z| < 2$ implies $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$, so we use appropriate MacLauren series. We have

$$\begin{aligned} f(z) &= -\frac{1}{2}\frac{1}{1 - \frac{z}{2}} - \frac{1}{z}\frac{1}{1 - \frac{1}{z}} \\ &= -\frac{1}{2}\sum_{n=0}^{\infty} \frac{z^n}{2^n} - \frac{1}{z}\sum_{n=0}^{\infty} \frac{1}{z^n} && (|1/z| < 1, \quad |z/2| < 1) \\ &= \sum_{n=0}^{\infty} (-1)\frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty} \frac{1}{z^n}. \end{aligned}$$

[1] However, it is **essential** that you understand how to solve each problem, so I **highly recommend** that you continue to work on any unfinished problems throughout the week. You can also post questions on the "quizzes/section" stream on Zulip: https://math103a.zulipchat.com.

(c) Since f is analytic on the annulus $A(0, 2, \infty)$, it has a Laurent series there. The annulus is centered at 0 - so we look for a Laurent series in powers of z. The condition $2 < |z| < \infty$ implies $\left|\frac{2}{z}\right| < 1$ and $\left|\frac{1}{z}\right| < \frac{1}{2} < 1$, which prompts us to use appropriate MacLauren series. We have

$$\begin{aligned} f(z) &= \frac{1}{z}\frac{1}{1-\frac{2}{z}} - \frac{1}{z}\frac{1}{1-\frac{1}{z}} \\ &= \frac{1}{z}\sum_{n=0}^{\infty}\frac{2^n}{z^n} - \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} && (|2/z| < 1, \quad |1/z| < 1) \\ &= \sum_{n=0}^{\infty}(2^n - 1)\frac{1}{z^{n+1}} \\ &= \sum_{n=1}^{\infty}(2^{n-1} - 1)\frac{1}{z^n}. \end{aligned}$$

(d) Since f is analytic on the annulus $A(1, 0, 1)$, it has a Laurent series there. But this annulus is centered at 1, so we seek a Laurent series involving powers of $z - 1$. The condition $|z - 1| < 1$ suggests which MacLauren series to use. We have

$$\begin{aligned} f(z) &= \frac{1}{-1 + (z-1)} - \frac{1}{z-1} \\ &= (-1)\frac{1}{1-(z-1)} - \frac{1}{z-1} \\ &= -\sum_{n=0}^{\infty}(z-1)^n - \frac{1}{z-1} && (|z-1| < 1). \end{aligned}$$

(e) Since f is analytic on the annulus $A(2, 0, 1)$, it has a Laurent series there. But this annulus is centered at 2, so we seek a Laurent series involving powers of $z - 2$. The condition $|z - 2| < 1$ suggests which MacLauren series to use. We have

$$\begin{aligned} f(z) &= \frac{1}{z-2} - \frac{1}{1+(z-2)} \\ &= \frac{1}{z-2} - \sum_{n=0}^{\infty}(-1)^n(z-2)^n && (|z-2| < 1). \end{aligned}$$