Quiz 5 (Solutions)

Directions: Complete as many problems as you can during section. This is a low-stakes "quiz" - you will not be penalized if you do not finish every problem.[1] You may collaborate with other members of the class, consult the TA, and consult the textbook or lecture notes. Scan and save your work as a `.pdf` file and submit via https://canvas.ucsc.edu. You have an additional 55 minutes after the meeting to scan and submit your work.

Q1. For $R > 0$, denote by C_R the upper half of the circle of radius R centered at the origin, with the positive orientation. Use the Triangle Inequality for Contour integrals to estimate the modulus of the following integrals.

(a) $\int_{C_2} \frac{\operatorname{Log} z}{z^2}\, dz$;

(b) $\int_{C_1} \frac{z^3}{z^5+3z+1}\, dz$.[2]

Solution. (a) If $z \in C_2$, then $|z| = 2$. Hence, $\left|z^2\right| = |z|^2 = 4$ and

$$\begin{aligned} |\operatorname{Log} z| &= |\ln |z| + i \operatorname{Arg} z| \\ &\leq |\ln 2| + |i|\,|\operatorname{Arg} z| \\ &\leq \ln 2 + \pi \qquad\qquad \text{(since } 0 < \operatorname{Arg} z < \pi \text{ on } C_2\text{)}. \end{aligned}$$

Hence, $\left|\frac{\operatorname{Log} z}{z^2}\right| \leq \frac{\ln 2+\pi}{4}$ on C_2. By the Triangle Inequality for Contour Integrals,

$$\left|\int_{C_2} \frac{\operatorname{Log} z}{z^2}\, dz\right| \leq \frac{\ln 2 + \pi}{4} \cdot 2\pi.$$

(b) If $z \in C_1$, then $|z| = 1$. Hence, $\left|z^3\right| = 1$ and

$$\left|z^5 + 3z + 1\right| \geq \left|\left|z^5 + 3z\right| - 1\right| = \left|z^5 + 3z\right| - 1 \geq \left|\left|z^5\right| - |3z|\right| - 1 = 1.$$

Note that we have used the fact that $\left|z^5 + 3z\right| - 1 \geq 0$ if $|z| = 1$. Hence,

$$\left|\frac{z^3}{z^5 + 3z + 1}\right| \leq 1$$

and C_1 and by the Triangle Inequality for Contour Integrals

$$\left|\int_{C_1} \frac{z^3}{z^5 + 3z + 1}\, dz\right| \leq \pi.$$

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Q2. Let C be the unit circle with positive orientation. For each function $f(z)$, determine whether you can conclude that

$$\int_C f(z)\, dz = 0$$

via the Cauchy-Goursat theorem.[3]

[1] However, it is **essential** that you understand how to solve each problem, so I **highly recommend** that you continue to work on any unfinished problems throughout the week. You can also post questions on the "quizzes/section" stream on Zulip: https://math103a.zulipchat.com.

[2] Note: There was a typo here on Monday, which made the problem too difficult to solve.

[3] Note: on Monday, this question was poorly worded and didn't make sense for (a) and (b). It has since been modified.

(a) $f(z) = \frac{z}{z^2+1}$;

(b) $f(z) = \operatorname{Log}(z+i)$;

(c) $f(z) = \tan z$;

(d) $f(z) = \frac{e^z}{\cosh z}$.

Solution. (a) This function is not analytic at $i, -i$, both of which lie on the unit circle. Hence, Cauchy-Goursat cannot be applied.

(b) This function is not analytic at many points. In particular, it is not analytic, or even defined, when $z = -i$. Since this point lies on C, the Cauchy-Goursat theorem cannot be applied.

(c) The zeros of $\cos z$ are $z = \pi/2 + k\pi i, k \in \mathbb{Z}$. None of these lie on or interior to the unit circle. Hence, $f(z)$ is analytic everywhere on and interior to C, so the Cauchy-Goursat theorem applies.

(d) The zeros of $\cosh z$ are $z = (\pi/2 + k\pi)i,\ k \in \mathbb{Z}$. None of these lie on or interior to the unit circle. Hence, $f(z)$ is analytic everywhere on and interior to C, so the Cauchy-Goursat theorem applies.

Q3. For each given function f and contour C, evaluate the integral

$$\int_C f(z)\, dz.$$

(a) $f(z)$ is the principal branch of the power function

$$z^{a-1} = e^{(a-1)\operatorname{Log} z},\ |z| > 0,\ -\pi < \operatorname{Arg} z < \pi,$$

a is a nonzero real number, and C is the positively oriented circle of radius R centered at the origin.

(b) $f(z)$ is the branch of the power function

$$z^i = e^{i \log z},\ |z| > 0,\ -\pi/2 < \operatorname{Arg} z < 3\pi/2.$$

and C is any contour that begins at $z = -i$ and ends at $z = i$ and lies to the right of the imaginary axis (except for its endpoints).

Solution. (a) The easiest way to solve this is by parameterize the circle. Here, I illustrate a method to compute the integral using antiderivatives. The contour crosses the branch cut, along which the function is not even defined. Therefore, the function has no antiderivative on any domain containing the entire curve. But, we can still integrate by cutting the curve into smaller pieces, and choosing an antiderivative on each piece.

Let C_1 denote the right half of the circle from $-Ri$ to Ri, let C_2 denote the left half of the circle from Ri to $-R$, and let C_3 denote the lower left half of the circle from $-R$ to $-Ri$. Then we have

$$\int_C f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz + \int_{C_3} f(z)\, dz.$$

On C_1, $f(z)$ is continuous and has an antiderivative, namely, the branch $F(z) = \frac{1}{a}z^a$, $(|z| > 0,\ -\pi < \operatorname{Arg} z < \pi)$. Hence,

$$\begin{aligned}\int_{C_1} z^{a-1}\, dz &= \frac{1}{a}\left(z^a\right)_{-iR}^{iR} \\ &= \frac{1}{a}((iR)^a - (-iR)^a) \\ &= \frac{R^a}{a}(e^{a\pi i/2} - e^{-a\pi i/2})\end{aligned}$$

On C_2, f isn't defined at $z = -1$ and hence not continuous. But the branch $g(z) = z^{a-1}$, $(|z| > 0, -\pi/2 < \arg z < 3\pi/2)$ agrees with f everywhere on C_2 and is continuous on C_2. Moreover, g has an antiderivative, $G(z) = \frac{1}{a}z^a$ $(|z| > 0, -\pi/2 < \arg z < 3\pi/2)$. Hence, we compute

$$\begin{aligned}\int_{C_2} f(z)\, dz &= \int_{C_2} g(z)\, dz \\ &= \frac{1}{a}(((-R)^a - (iR)^a))) \\ &= \frac{R^a}{a}(e^{a\pi i} - e^{a\pi i/2}).\end{aligned}$$

For similar reasons, we choose the branch $h(z) = z^{a-1}$, $(|z| > 0, -2\pi < \arg z < 0)$ which agrees with f everywhere on C_3. We have

$$\begin{aligned}\int_{C_3} f(z)\, dz &= \int_{C_3} h(z)\, dz \\ &= \frac{1}{a}((-iR)^a - (-R)^a) \\ &= \frac{R^a}{a}(e^{-a\pi i/2} - e^{-a\pi i}).\end{aligned}$$

Evidently, then

$$\int_C f(z)\, dz = \frac{R^a}{a}(e^{a\pi i} - e^{-a\pi i}) = 2i\frac{R^a}{a}\sin a\pi.$$

(b) The integral exists since $f(z)$ is piecewise continuous on C. However, the $f(z)$ is not even defined at $z = -i$ and hence not continuous on any domain containing C. Therefore, f has no antiderivative on any domain containing the curve. However, we can replace the integrand with a different branch of the power function that does have an antiderivative, provided that the two branches agree everywhere on C. Consider the branch $g(z) = z^i$, $|z| > 0,\ -\pi < \arg z < \pi$. Then f and g agree at every point on C and g is continuous on C, and hence in some domain containing C. Since g has an antiderivative, $G(z) = \frac{1}{i+1}z^{i+1} = \frac{i-1}{2}z^{i+1}$, $(|z| > 0,\ -\pi < \arg z < \pi)$, on such a domain, we can evaluate the integral as follows:

$$\begin{aligned}\int_C f(z)\, dz &= \int_C g(z)\, dz \\ &= \frac{i-1}{2}(i^{i+1} - (-i)^{i+1}) \\ &= \frac{i-1}{2}(e^{(i+1)\pi/2} - e^{-(i+1)\pi/2}) \\ &= (i-1)\sinh((i+1)\pi/2)\end{aligned}$$