Quiz 4 + Solutions

Directions: Complete as many problems as you can during section. This is a low-stakes "quiz" - you will not be penalized if you do not finish every problem.[1] You may collaborate with other members of the class, consult the TA, and consult the textbook or lecture notes. Scan and save your work as a `.pdf` file and submit via https://canvas.ucsc.edu. You have until midnight to scan and submit your work.

Q1. Evaluate the contour integral $\int_C f(z)\,dz$ where $f(z) = \pi e^{\pi \overline{z}}$ and C is the positively oriented boundary of the square with vertices at $0, 1, 1+i, i$.

Solution.

Parameterize each side of the square as follows: $C_1 : z_1(t) = t,\ t \in [0,1],\ C_2 : z_2(t) = 1+it,\ t \in [0,1],\ C_3 : z_3(t) = (1-t)+i,\ t \in [0,1]$, and $C_4 : z_4(t) = i(1-t),\ t \in [0,1]$. Then

$$
\begin{aligned}
\int_C f(z)\,dz &= \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz + \int_{C_4} f(z)\,dz \\
&= \pi\left(\int_0^1 e^{\pi t}\,dt + i\int_0^1 e^{\pi(1-it)}\,dt - \int_0^1 e^{\pi((1-t)-i)}\,dt - i\int_0^1 e^{\pi i(t-1)}\,dt\right) \\
&= \pi\left(\int_0^1 e^{\pi t}\,dt + ie^{\pi}\int_0^1 e^{-\pi it}\,dt - e^{\pi - i\pi}\int_0^1 e^{-\pi t}\,dt - ie^{-\pi i}\int_0^1 e^{\pi it}\,dt\right) \\
&= \pi\left(\frac{1}{\pi}(e^{\pi}-1) - \frac{ie^{\pi}}{\pi i}(e^{-\pi i}-1) + \frac{e^{\pi-\pi i}}{\pi}(e^{-\pi}-1) - \frac{ie^{-\pi i}}{\pi i}(e^{\pi i}-1)\right) \\
&= (e^{\pi}-1) - e^{\pi}((-1)-1) + -e^{\pi}(e^{-\pi}-1) + ((-1)-1) \\
&= 4(e^{\pi}-1).
\end{aligned}
$$

□

Q2. Evaluate the contour integral $\int_C f(z)\,dz$ where $f(z) = z + \overline{z}$ and C is the negatively oriented circle of radius 2, centered at 0.

Solution. Parameterize the circle as $z(t) = 2e^{-it}$ for $0 \le t \le 2\pi$. Then $z'(t) = -2ie^{-it}$. Hence,

$$
\begin{aligned}
\int_C z + \overline{z}\,dz &= \int_0^{2\pi} (2e^{-it} + 2e^{it})(-2ie^{-it})\,dt \\
&= 4i\int_0^{2\pi} e^{-it}e^{-it}\,dt - 4i\int_0^{2\pi} e^{it}e^{-it}\,dt \\
&= 4i\int_0^{2\pi} e^{-2it}\,dt - 4i\int_0^{2\pi} 1\,dt \\
&= 4i\left[\frac{1}{-2i}e^{-2it}\right]_0^{2\pi} - 8\pi i \\
&= 4i\left(\frac{1}{-2i}e^{-4\pi i} - \frac{1}{-2i}\right) - 8\pi i \\
&= -8\pi i.
\end{aligned}
$$

□

[1] However, it is **essential** that you understand how to solve each problem, so I **highly recommend** that you continue to work on any unfinished problems throughout the week. You can also post questions on the "quizzes/section" stream on Zulip: https://math103a.zulipchat.com.

Q3. Evaluate the contour integral $\int_C f(z)\,dz$ where $f(z) = \operatorname{Re} z$ and C is the part of the parabola $y = x^2$ joining 0 to $1 + i$.

Solution. Parameterize C as $z(t) = t + it^2$ for $t \in [0, 1]$. Then $z'(t) = (1 + 2it)$. Hence,

$$\begin{aligned}
\int_C \operatorname{Re} z\,dz &= \int_0^1 \operatorname{Re}(t + it^2)(1 + 2it)\,dt \\
&= \int_0^1 t + 2it^2\,dt \\
&= \int_0^1 t\,dt + 2i \int_0^1 t^2\,dt \\
&= \frac{1}{2} + \frac{2}{3}i.
\end{aligned}$$

□