Quiz 3

Directions: Complete as many problems as you can during section. This is a low-stakes "quiz" - you will not be penalized if you do not finish every problem.[1] You may collaborate with other members of the class, consult the TA, and consult the textbook or lecture notes. Scan and save your work as a `.pdf` file and submit via https://canvas.ucsc.edu. You have an additional 55 minutes after the meeting to scan and submit your work.

Q1. Find all values of the following expressions and state which value is principal. (The *principal value* of a multiple valued expression is the value that occurs when the principal branch of the defining function is used, for example, $\operatorname{Log} i$ is the principal value of $\log i$.)

(a) i^{-i};

(b) 1^i;

(c) $(1+i)^{2i}$.

Solution. (a) Note that $\arg(-i) = -\pi/2 + 2k\pi$, $k \in \mathbb{Z}$, and $|-i| = 1$. Hence, we have

$$\begin{aligned} i^{-i} &= \exp(i \log(-i)) \\ &= \exp(i \ln|-i| + i^2 \arg(-i)) \\ &= \exp(\pi/2 - 2k\pi),\ k \in \mathbb{Z}. \end{aligned}$$

The principal value of log occurs when $k = 0$. Hence, $\mathrm{P.V.}\, i^i = e^{\pi/2}$.

(b) Since $\arg 1 = 2k\pi$ and $|1| = 1$, we have

$$\begin{aligned} 1^i &= \exp(i \log 1),\ k \in \mathbb{Z} \\ &= e^{-2k\pi},\ k \in \mathbb{Z} \end{aligned}$$

and evidently $\mathrm{P.V.}\, 1^i = e^0 = 1$.

(c) Note that $\arg(1+i) = \pi/4 + 2k\pi$, $k \in \mathbb{Z}$ and $|1+i| = \sqrt{2}$. So we have

$$\begin{aligned} (1+i)^{2i} &= \exp(2i \ln\sqrt{2} - 2(\pi/4 + 2k\pi)) \\ &= \exp(i \ln 2 - 2\pi(8k+1)/4) \\ &= e^{-\pi(8k+1)/2}(\cos\ln 2 + i \sin\ln 2). \end{aligned}$$

Moreover, $\mathrm{P.V.}(1+i)^{2i} = e^{-\pi/2}(\cos\ln 2 + i\sin\ln 2)$.

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Q2. Find all $z \in \mathbb{C}$ satisfying each equation:

(a) $(e^z)^2 - (2+2i)e^z + 2i = 0$.[2]

(b) $\cos z = i$. [3]

Solution. (a) This factors so that $(e^z - (1+i))^2 = 0$. Hence. $e^z = 1+i$. Solutions to this equation are $z = \log(1+i) = \ln\sqrt{2} + i(\pi/4 + 2k\pi)$.

[1] However, it is **essential** that you understand how to solve each problem, so I **highly recommend** that you continue to work on any unfinished problems throughout the week. You can also post questions on the "quizzes/section" stream on Zulip: https://math103a.zulipchat.com.

[2] Hint: PSet 1 P1.

[3] Hint: use the definition of cosine, and try to solve for z. Rewrite the equation as a "quadratic in e^z" as in (a).

(b) If you followed the hint, or else payed attention in the lecture (depending on which day you took the quiz) you will have found that

$$\begin{aligned} z = \cos^{-1} i &= -i \log(i + i(1 - i^2)^{1/2}) \\ &= -i \log(i \pm \sqrt{2}i). \end{aligned}$$

The values of $\log(i + \sqrt{2}i)$ are

$$\ln(1+\sqrt{2}) + i(\pi/2 + 2k\pi) = \ln(1+\sqrt{2}) + \frac{\pi i}{2}(4k+1), \ k \in \mathbb{Z}$$

and the values of $\log(i - \sqrt{2}i)$ are

$$\ln(\sqrt{2}-1) + i(3\pi/2 + 2l\pi) = \ln(\sqrt{2}-1) + \frac{\pi i}{2}(4l+3), \ l \in \mathbb{Z}.$$

Notice that $\ln(\sqrt{2}-1) = -\ln(1+\sqrt{2})$. Notice that as k, l range over $\mathbb{Z}$, $4k+1, 4l+3$ (together) range over the odd integers. So we can describe write

$$-i\log(i \pm \sqrt{2}i) = -i((-1)^n \ln(1+\sqrt{2}) + \frac{\pi i}{2}(2n+1)) = (-1)^{n+1} i \ln(1+\sqrt{2}) - \frac{\pi}{2}(2n+1), \ n \in \mathbb{Z}.$$

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Q3. Determine the singular points of each function:

(a) $f(z) = \dfrac{2z+1}{z^3 - i}$;

(b) $f(z) = |z|^2$;

(c) $f(z) = \frac{z^2}{\cos(z) - i}$.

Solution. (a) Since f is a rational function, it is analytic everywhere except when $z^3 = i$. The solutons to this equation are $z_i = e^{i(\pi/6 + 2k\pi/3)}$, $k = 0, 1, 2$. Clearly, f is analytic at a point in every neighborhood of each of these points.

(b) There are no singular points. If you check the Cauchy-Riemann equation, the only solutions is 0 and in fact the derivative does exist here (check using the definition). But f can't be analytic at 0 since every neighborhood of 0 has points at which f is not analytic.

(c) The singular points are the values found in Q2 (b).

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Q4. Show that each statement is false by giving a counterexample:

(a) $\operatorname{Log} z + \operatorname{Log} w = \operatorname{Log} zw$;

(b) Using the principal branch of log, $|z|^a = |z^a|$ for any $a \in \mathbb{C}$.

Solution. (a) Take $z = w = -1$ to see that $\operatorname{Log} zw = \operatorname{Log} 1 = 0$ while $\operatorname{Log} z + \operatorname{Log} w = 2\operatorname{Log} -1 = 2\pi i$.

(b) Let e denote Euler's number (a real number) and let e^w denote the complex exponential function, which is defined via $e^w := e^u(\cos v + i \sin v)$ where $w = u + iv$. Then

$$|e|^{\pi i} = e^{\pi i} = e^{\pi i \operatorname{Log} e} = e^{\pi i(\ln e + \operatorname{Arg} e)} = e^{\pi i} = \cos\pi + i\sin\pi = -1.$$

On the other hand,

$$\left|e^{\pi i}\right| = \left|e^{\pi i}\right| = |-1| = 1.$$

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