Quiz 2 (Solutions)

Directions: Complete as many problems as you can during section. This is a low-stakes "quiz" - you will not be penalized if you do not finish every problem.[1] You may collaborate with other members of the class, consult the TA, and consult the textbook or lecture notes. Scan and save your work as a `.pdf` file and submit via https://canvas.ucsc.edu. You have an additional 55 minutes after the meeting to scan and submit your work.

Q1. For each function $f(z)$, find real valued functions $u(x,y)$ and $v(x,y)$ such that $f(x+iy) = u(x,y) + iv(x,y)$

(a) $f(z) = z^2 + z + 1$.

(b) $f(z) = \dfrac{\overline{z}^2}{z}$, $z \neq 0$.[2]

Proof. (a) Writing $z = x + iy$, we have

$$\begin{aligned} f(x+iy) &= (x+iy)^2 + x + iy + 1 \\ &= x^2 + 2ixy - y^2 + x + iy + 1 \\ &= (x^2 - y^2 + x + 1) + i(2xy + y) \end{aligned}$$

so that $u(x,y) = x^2 - y^2 + x + 1$ and $v(x,y) = 2xy + y$.

(b) Note that $\frac{\overline{z}^2}{z} = \frac{\overline{z}^3}{|z|^2}$. We can thus use the binomial theorem to make the computation easier. We have

$$\begin{aligned} f(x+iy) &= \frac{(x-iy)^3}{x^2+y^2} \\ &= \frac{x^3 - 3ix^2y - 3xy^2 + iy^3}{x^2+y^2} \\ &= \frac{x^3 - 3xy^2}{x^2+y^2} + i\frac{-3x^2y + y^3}{x^2+y^2} \end{aligned}$$

so that $u(x,y) = \frac{x^3-3xy^2}{x^2+y^2}$ and $v(x,y) = \frac{-3x^2y+y^3}{x^2+y^2}$.

Q2. Compute the derivative at $z_0 \in \mathbb{C}$, if it exists. You may use either the limit definition of the derivative, or the necessary/sufficient conditions involving the Cauchy-Riemann equations.

(a) $f(z) = \overline{z}$;

(b) $f(z) = z^2 + z + 1$.

Solution. Both parts can be solved using the limit definition, but I will use the CR-equations.

(a) Writing $z = x+iy$ we see that $f(z) = x - iy$ so that $u(x,y) = x$ and $v(x,y) = -y$. Hence, $u_x = 1$ while $v_y = -1$. Evidently then, $u_x(x,y) \neq v_y(x,y)$ for any point $(x,y) \in \mathbb{R}^2$. Thus, the CR-equations are satisfied nowhere and so f is differentiable nowhere.

[1] However, it is **essential** that you understand how to solve each problem, so I **highly recommend** that you continue to work on any unfinished problems throughout the week. You can also post questions on the "quizzes/section" stream on Zulip: https://math103a.zulipchat.com.

[2] cf. Problem set 2.

(b) Writing $z = x + iy$ we see that

$$f(z) = (x+iy)^2 + (x+iy) + 1 = x^2 - y^2 + x + 1 + i(2xy + y)$$

so that $u(x,y) = x^2 - y^2 + x + 1$ and $v(x,y) = 2xy + y$. Hence, $u_x = 2x + 1$, $u_y = -2y$, $v_x = 2y$, and $v_y = 2x + 1$. Hence, $u_x = v_y$ and $u_y = -v_x$ so the CR-equations are satisfied at every point in $\mathbb{R}^2$. Moreover, the first order partials of u, v are continuous on $\mathbb{R}^2$. Hence, f is entire and

$$f'(z) = u_x(x,y) + iv_x(x,y) = 2x + 1 + i2y = 2(x+iy) + 1 = 2z + 1.$$

Q3. Show that the limit does not exist:[3]

$$\lim_{z \to 0} \frac{z^2}{\overline{z}^2}.$$

Proof. Along the real axis, we have

$$\begin{aligned} \lim_{z \to 0} \frac{z^2}{\overline{z}^2} &= \lim_{x \to 0} \frac{x^2}{x^2} \\ &= 1. \end{aligned}$$

But along the line $y = x$, we see that

$$\begin{aligned} \lim_{z \to 0} \frac{z^2}{\overline{z}^2} &= \lim_{x \to 0} \frac{(x+ix)^2}{(x-ix)^2} \\ &= \lim_{x \to 0} \frac{(1+i)^2}{(1-i)^2} \\ &= -1. \end{aligned}$$

So the limit does not exist.

Q4. Compute the following limits using the theorem on limits involving the point at infinity:

(a) $\displaystyle \lim_{z \to \infty} \frac{9z^2}{(2z-1)^2}$;

(b) $\displaystyle \lim_{z \to 2} \frac{1}{(z-2)^2}$;

(c) $\displaystyle \lim_{z \to \infty} \frac{z^2+1}{z-1}$.

Proof. (a) It looks like this limit will be finite. So we use (2) on page 51 in the book. We have

$$\begin{aligned} \lim_{z \to 0} \frac{9/z^2}{(2/z-1)^2} &= \lim_{z \to 0} \frac{9z^2}{z^2(2-z)^2} \\ &= \lim_{z \to 0} \frac{9}{(2-z)^2} \\ &= 9/4. \end{aligned}$$

Hence, $\displaystyle \lim_{z \to \infty} \frac{9z^2}{(2z-1)^2} = 9/4$.

[3]Hint: Look at the value of $\frac{z^2}{\overline{z}^2}$ along the real axis and ~~imaginary axes~~ the line $y = x$.

(b) It looks like this limit will be infinite. So we use (1) on page 50 in the book. We have

$$\begin{aligned}\lim_{z\to 2} \frac{1}{\frac{1}{(z-2)^2}} &= \lim_{z\to 2}(z-2)^2 \\ &= 0.\end{aligned}$$

Hence, $\lim\limits_{z\to 2} \dfrac{1}{(z-2)^2} = \infty$.

(c) It looks like this limit will be infinite. So we use (3) on page 51 in the book. We have

$$\begin{aligned}\lim_{z\to 0} \frac{1}{\frac{1/z^2+1}{1/z-1}} &= \lim_{z\to 0} \frac{1/z-1}{1/z^2+1} \\ &= \lim_{z\to 0} \frac{(1-z)z^2}{z(1+z^2)} \\ &= \lim_{z\to 0} \frac{(1-z)z}{(1+z^2)} \\ &= 0.\end{aligned}$$

Hence, $\lim\limits_{z\to\infty} \dfrac{z^2+1}{z-1} = \infty$.

Q5. Using the Cauchy-Riemann equations, determine all points at which the following functions are differentiable:

(a) $f(z) = \operatorname{Re} z$;

(b) $f(z) = \operatorname{Im} z$.

Proof. Write $z = x + iy$ and $z_0 = x_0 + iy_0$.

(a) The difference quotient is

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{x - x_0}{(x - x_0) - i(y - y_0)}$$

Evidently the value of this expression is 0 along the line $x = x_0$. But along the line $y = y_0$, the value is 1. Hence, the limit $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ does not exist. Since z_0 was arbitrary, this function is not differentiable anywhere.

Alternatively, notice that $u = x$ and $v = 0$ so that $u_x = 1 \neq 0 = v_y$. Hence, the CR-equations are satisfied nowhere and f is nowhere differentiable.

(b) The difference quotient is

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{y - y_0}{(x - x_0) - i(y - y_0)}$$

Similarly to (a), the value of this expression is 0 along the line $y = y_0$. But along the line $x = x_0$, the value is i.

Alternatively, notice that $u = 0$ and $v = y$ so that $u_x = 0 \neq 1 = v_y$. Hence, the CR-equations are satisfied nowhere and f is nowhere differentiable.