Quiz 1 + Solutions

Directions: Complete as many problems as you can during section. This is a low-stakes "quiz" - you will not be penalized if you do not finish every problem.[1] You may collaborate with other members of the class, consult the TA, and consult the textbook or lecture notes. Scan and save your work as a `.pdf` file and submit via https://canvas.ucsc.edu. You have an additional 55 minutes after the meeting to scan and submit your work.

Q1. Compute the following complex numbers. Express your solution in the form $x + iy$.

(a) $z_1 = \dfrac{7-3i}{2i+3}$

(b) $z_2 = \dfrac{(-\sqrt{3}i-1)^{30}}{(-1-i)^{60}}$.

(c) $\operatorname{Re}\left(i + \frac{1}{i}\right)$.

Proof. (a) We saw in class that $\dfrac{1}{w} = \dfrac{\overline{w}}{|w|^2}$. Hence, we have

$$\begin{aligned}\frac{7-3i}{2i+3} &= \frac{(7-3i)(3-2i)}{(3+2i)(3-2i)} \\ &= \frac{(21-6)-(14+9)i}{13} \\ &= \frac{15}{13} - \frac{23}{13}i.\end{aligned}$$

(b) Use polar coordinates. We have $\left|-\sqrt{3}i-1\right| = 2$ and $\operatorname{Arg}(-\sqrt{3}i-1) = -\frac{2\pi}{3}$ so that $-\sqrt{3}i - 1 = 2e^{-\frac{2\pi}{3}i}$. Similarly, $-1-i = \sqrt{2}e^{-\frac{3\pi}{4}i}$. Thus,

$$\begin{aligned}\frac{(-\sqrt{3}i-1)^{30}}{(-1-i)^{60}} &= \frac{2^{30}e^{-20\pi i}}{\sqrt{2}^{60}e^{-45\pi i}} \\ &= e^{25\pi i} \\ &= -1.\end{aligned}$$

☕

Q2. Find all distinct solutions the equation $z^3 = -2$. Express them in the form $x + iy$.

Proof. Write $z = re^{i\theta}$. Then the equation becomes $r^3e^{3i\theta} = 2e^{\pi i}$. Hence. $r = \sqrt[3]{2}$ and $\theta = \frac{2k+1}{3}\pi$, $k = 0, 1, 2$. So there are three solutions $z_1 = \sqrt[3]{2}e^{i\pi/3} = \sqrt[3]{2}\left(1/2 + i\sqrt{3}/2\right)$, $z_2 = -\sqrt[3]{2}$, and $z_3 = \sqrt[3]{2}\left(1/2 - i\sqrt{3}/2\right)$. ☕

Q3. Sketch the following sets. For each set, determine which points are interior, and which points are boundary points (no proof needed).

(a) $\{z \in \mathbb{C} : |z-i| = |z+i|\}$;

(b) $\{z \in \mathbb{C} : |z| > 1 - \operatorname{Re} z\}$;

[1] However, it is **essential** that you understand how to solve each problem, so I **highly recommend** that you continue to work on any unfinished problems throughout the week. You can also post questions on the "quizzes/section" stream on Zulip: https://math103a.zulipchat.com.

(c) $\{z \in \mathbb{C} : |2z + 3| > 4\}$;

(d) $\{z \in \mathbb{C} : \mathrm{Re}(z(1 - i)) < 2\}$;

(e) $\{z \in \mathbb{C} \setminus \{0\} : 0 < \arg z < \pi/3\} \cup \{0\}$.

(f) $\mathbb{C} \setminus \{z \in \mathbb{C} : 0 < |z| < 1\}$.

Proof. (a) This is the set of points equidistant from i and $-i$, that is, the real line. The interior is empty and every point is a boundary point.

(b) This is the set of points lying "outside" of the parabola $y^2 + 2x - 1$. Every point $x + iy$ with $y^2 + 2x - 1 > 0$ is interior (i.e. the set is open c.f. PSet 1 #8) and every point on the parabola is a boundary point.

(c) This is the complement of a closed disk of radius 2, centered at $-3/2$. Every point in the set is interior (i.e. the set is open) and the boundary is the circle $C_2(-3/2)$.

(d) Th condition simplifies to $x + y < 2$ which is the set of points lying on one side of a line. Every point is interior and the boundary is the line.

(e) This looks like an unbounded slice of π.[2] The every point in the set is interior except 0. The boundary points are the rays $\theta = 0, r > 0$ and $\theta = \pi/3, r > 0$, and 0.

(f) This is the complement of the open disk $D_1(0)$ together with the point 0. The interior is the complement of the closed disk $\overline{D_1(0)}$. The boundary points are 0 and the points on the circle $C_1(0)$.

☕

Q4. Determine whether the following statements are true or false. Prove the true statements. Disprove the false statements by constructing a counterexample.

(a) $\mathrm{Arg}\, z_1 - \mathrm{Arg}\, z_2 = \mathrm{Arg}\, z_1/z_2$.

(b) z is real if and only if $\overline{z} = z$.

(c) Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial with complex coefficients. If $p(\zeta) = 0$, then $p(\overline{\zeta}) = 0$.

Proof. (a) This is false. Take for example $z_1 = i$ and $z_2 = -1 - i$. Then $\mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2 = \pi/2 - (-3\pi/4) = 5\pi/4$. But $\frac{z_1}{z_2} = \frac{-1-i}{2}$ so that $\mathrm{Arg}\, \frac{z_1}{z_2} = -3\pi/4$.

(b) This is true. The forward implication is obvious by definition of the conjugate. Conversely, say $z = x + iy$ and $z = \overline{z}$. Then $x + iy = x - iy$ so that $y = -y$. Since $y \in \mathbb{R}$, $y = 0$. Thus, $z = x \in \mathbb{R}$.

(c) This is false. Take $p(z) = (z - i)^2$. Clearly, i is a root, but $\overline{i} = -i$ is not. Howoever, if $p(z)$ has *real* coefficients, the statement is true: if $p(\zeta) = 0$, then

$$p(\overline{\zeta}) = \sum_{i=0}^{n} a_i \overline{\zeta}^i = \sum_{i=0}^{n} \overline{a_i}\,\overline{\zeta^i} = \sum_{i=0}^{n} \overline{a_i \zeta^i} = \overline{\sum_{i=0}^{n} a_i \zeta^i} = \overline{p(\zeta)} = 0.$$

☕

[2] kind of joking here