Weekly Assignment 2
Selected Solutions

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MATH 117: Advanced Linear Algebra

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Some hints for this assignment are written in the footnotes. See the weekly assignment webpage for due dates, templates, and assignment description. Make sure to justify any claims you make. You may not appeal to any results that we have not discussed in class.

1. Suppose that $V = U \oplus W$ is the direct sum of subspaces U and W. The projection onto U along W is the function $P : V \to V$ defined via $P(v) = u$ where $u \in U$ is the unique vector satisfying $v = u + w$ for some $w \in W$.

(a) Prove that P is linear.

(b) Prove that a linear operator $L : V \to V$ is a projection onto some subspace if and only if $L^2 = L$.

Proof. (a) Let $v, v' \in V$. Then there exist unique vectors $u, u' \in U$ and $w, w' \in W$ such that $v = u + w$ and $v' = u' + v'$. Hence,

$$P(v + v') = P((u + w) + (u' + w')) = P((u + u') + (w + w')) = u + u' = P(v) + P(v')$$

where the second-to-last equality follows from the fact that $u + u' \in U$ and $w + w' \in W$. Let $\alpha \in F$. Then $\alpha u \in U$ and $\alpha w \in W$. Hence,

$$P(\alpha v) = P(\alpha u + \alpha w) = \alpha u = \alpha P(v).$$

This proves the claim in (a).

(b) Suppose that L is a projection onto a subspace U of V. Let W be a complement of U in V. Let $v \in v$. Then $v = u + w$ for unique $u \in U$ and $w \in W$. Since L is a projection onto U, we have $L(v) = u$ and $L(u) = u$. Hence,

$$L^2(v) = L(u) = u = L(v).$$

Conversely, suppose that $L^2 = L$. First, we claim that $V = \ker L \oplus \operatorname{im} L$. Let $v \in \ker L \cap \operatorname{im} L$. Then $v = L(v')$ for some $v' \in V$ and $L(v) = 0$. Hence, $0 = L(v) = L^2(v') = L(v') = v$. Thus, $\ker L \cap \operatorname{im} L = \{0\}$. Let $v \in V$. Then $v = (v - L(v)) + L(v)$. Clearly, $L(v) \in \operatorname{im} L$ and $v - Ł(v) \in \ker L$ since $L(v - L(V)) = L(v) - L^2(v) = L(v) - L(v) = 0$. This proves that $V = \ker L + \operatorname{im} L$. We conclude that $V = \ker L \oplus \operatorname{im} L$. It follows immediately that L is projection onto $\operatorname{im} L$ along $\ker L$.

$\square$

2. Let V_1, V_2 be vector spaces over F. The direct sum $V_1 \oplus V_2$ comes with linear maps

$$\iota_1 : V_1 \to V_1 \oplus V_2,\ v_1 \mapsto (v_1, 0) \quad \text{and} \quad \iota_2 : V_2 \to V_1 \oplus V_2,\ v_2 \mapsto (0, v_2).$$

Let Z be any other vector space and let $L_1 : V_1 \to Z$ and $L_2 : V_2 \to Z$ be any other linear maps. Prove that there is a unique linear map $L : V_1 \oplus V_2 \to Z$ with the property that $L \circ \iota_1 = L_1$ and $L \circ \iota_2 = L_2$. See Remark 1.

Proof. Define $L : V_1 \oplus V_2 \to Z$ via the rule $L(v_1, v_2) = L_1(v_1) + L_2(v_2)$. It is straightforward to verify that L is a linear map. Moreover, L has the desired property:

$$(L \circ \iota_1)(v_1) = L((v_1, 0)) = L(v_1)$$

for all $v_1 \in V_1$ and similarly $L \circ \iota_2 = L_2$. This proves existence. Suppose $K : V_1 \oplus V_2 \to Z$ is another linear map satisfying the given condition. Then for any $(v_1, v_2) \in V_1 \oplus V_2$, we have

$$K(v_1, v_2) = K(\iota_1(v_1) + \iota_2(v_2)) = (K \circ \iota_1)(v_1) + (K \circ \iota_2)(v_2) = L_1(v_1) + L_2(v_2) = L(v_1, v_2).$$

Thus, $K = L$ which proves that the map L is unique. $\square$

3. Let V_1, V_2, W_1, W_2 be vector spaces over F and let $L_1 : V_1 \to W_1$ and $L_2 : V_2 \to W_2$ be linear maps.

(a) Use the Universal Property of the Direct Sum (see Remark 1) to show that there is a unique linear map

$$L_1 \oplus L_2 : V_1 \oplus V_2 \to W_1 \oplus W_2$$

satisfying $(L_1 \oplus L_2)(v_1, v_2) = (L_1(v_1), L_2(v_2))$.

(b) Suppose additionally that V_1, V_2, W_1, W_2 are finite-dimensional with ordered bases $B_1 = (a_1, \ldots, a_k)$, $B_2 = (b_1, \ldots, b_l)$, $C_1 = (c_1, \ldots, c_m)$, and $C_2 = (d_1, \ldots, d_n)$, respectively.

(i) Prove that $B := ((a_1, 0), \ldots, (a_k, 0), (0, b_1), \ldots, (0, b_l))$ is a basis for $V_1 \oplus V_2$. Similarly, $C := ((c_1, 0), \ldots, (c_l, 0), (0, d_1), \ldots, (0, d_n))$ is a basis for $W_1 \oplus W_2$.

(ii) Prove that the matrix for $L_1 \oplus L_2$ with respect to B and C has the following block diagonal form:

$$[L_1 \oplus L_2]_B^C = \begin{pmatrix} [L_1]_{B_1}^{C_1} & 0 \\ 0 & [L_2]_{B_2}^{C_2} \end{pmatrix}.$$

Proof. (a) Let $\iota_{V_i} : V_i \to V_1 \oplus V_2$ and $\iota_{W_i} : W_i \to W_1 \oplus W_2$, $i \in \{1, 2\}$, denote the inclusions. Then we have linear maps $\iota_{W_i} \circ L_i : V_i \to W_1 \oplus W_2$ for $i \in \{1, 2\}$. According the the Universal Property of the Direct Sum, there is a unique linear map $L_1 \oplus L_2 : V_1 \oplus V_2 \to W_1 \oplus W_2$ satisfying $(L_1 \oplus L_2) \circ \iota_{V_1} = \iota_{W_1} \circ L_1$ and $(L_1 \oplus L_2) \circ \iota_{V_2} = \iota_{W_2} \circ L_2$. We just need to show that $(L_1 \oplus L_2)(v_1, v_2) = (L_1(v_1), L_2(v_2))$. For any $(v_1, v_2) \in V_1 \oplus V_2$, we have

$$\begin{aligned}
(L_1 \oplus L_2)(v_1, v_2) &= (L_1 \oplus L_2)(\iota_{V_1}(v_1) + \iota_{V_2}(v_2)) \\
&= (L_1 \oplus L_2)(\iota_{V_1}(v_1)) + (L_1 \oplus L_2)(\iota_{V_2}(v_2)) \\
&= (\iota_{W_1} \circ L_1)(v_1) + (\iota_{W_1} \circ L_1)(v_2) \\
&= (L_1(v_1), L_2(v_2)).
\end{aligned}$$

This proves the claim.

(b) To prove (i), it suffices to show that B is a spanning set (or that it is independent) because $|B| = k + l = \dim V_1 + \dim V_2 = \dim V_1 \oplus V_2$. It is straight forward to show that B is a spanning set. To prove (ii), one could show that the matrices have the same columns, i.e., compute coordinate vectors. I am omitting a proof of this.

$\square$

4. Let V be finite-dimensional vector space and W a subspace. Suppose that $\{b_1, \ldots, b_k\}$ is a basis for W and extend this to a basis $\{b_1, \ldots, b_k, b_{k+1}, \ldots, b_n\}$ for V using Proposition 1.4.11. Prove that the set of vectors $\{b_{k+1} + W, \ldots, b_n + W\}$ is a basis for the quotient space V/W.

Proof. We know that $V \cong W \oplus V/W$ so that $\dim V/W = \dim V - \dim W = (n - k) = |\{b_{k+1} + W, \ldots, b_n + W\}|$. Thus, it suffices to show that $\{b_{k+1} + W, \ldots, b_n + W\}$ is a spanning set for (equivalently, an independent subset of) V/W. Let $v \in V$. Then there are coefficient, $\alpha_1, \ldots, \alpha_n \in F$ such that $v = \sum_{i=1}^{n} \alpha_i b_i$. Hence,

$$\begin{aligned}
v + W &= \sum_{i=1}^{n} \alpha_i b_i + W \\
&= \sum_{i=1}^{k} \alpha_i b_i + W + \sum_{i=k+1}^{n} \alpha_i b_i + W \\
&= W + \sum_{i=k+1}^{n} \alpha_i b_i + W \\
&= \sum_{i=k+1}^{n} \alpha_i b_i + W
\end{aligned}$$

where we have used the fact that $\sum_{i=1}^{k} \alpha_i b_i + W = W$ since $\sum_{i=1}^{k} \alpha_i b_i \in W$. This proves that the set spans V/W, completing the proof. $\square$

Remark 1. *In other words, L is the unique linear map making the following diagram commute*

$$\begin{tikzcd}
V_1 \arrow[dr, "\iota_1"'] \arrow[drr, bend left, "L_1"] & & \\
& V_1 \oplus V_2 \arrow[r, dashed, "\exists! L"] & Z \\
V_2 \arrow[ur, "\iota_2"] \arrow[urr, bend right, "L_1"'] & &
\end{tikzcd}$$

The property described in Problem 1 is usually called the Universal Property of the Direct Product. It provides a precise answer to the question "How do I define a linear map out of the direct sum of two vector spaces?". One simply defines linear maps L_1 and L_2 as in the statement. Your proof will provide the recipe for constructing the desired linear map L. Moreover, it establishes a bijection of sets

$$\begin{aligned}
\mathrm{Hom}_F(V_1 \oplus V_2, Z) &\longleftrightarrow \mathrm{Hom}_F(V_1, Z) \oplus \mathrm{Hom}_F(V_2, Z) \\
L &\longmapsto (L \circ \iota_1, L \circ \iota_2)
\end{aligned}$$

However, notice that the domain and codomain are actually vector spaces. It can be shown that this bijection is a linear map, i.e., a vector space isomorphism. Note that all of this readily generalizes to the direct sum of finitely many vector spaces. Compare all of this with the discussion on the Universal Property of the Quotient Space from the lecture.