Midterm Exam Solutions

Overview: This exam consists of one cover page and **three pages of questions**. There are **7 questions** worth a total of **117 points**. Your grade on this exam will be calculated as

(**number of points earned**)/100.

Therefore, you need not feel pressured to complete every problem (but if you do, you can earn some extra credit). This exam is worth **15% of your final grade** in the course.

Directions:

- Solve as many of the following problems as you can in 90 minutes. Try to start with problems that you know how to solve right away, and save the others for last.
- You are **required** to provide sufficient justification for **every** problem you submit. You should show all of your computations and explain in detail your reasoning. If you do not provide sufficient justification, as judged solely by the grader, you will receive **NO CREDIT** for that problem, even if your answer is correct.
- Submit your work via the Midterm Exam quiz on Canvas. **You will only be able to submit 1 file and it MUST be a .pdf.** You can use CombinePDF to combine your files into a single .pdf for free.
- Read the exam protocol in the syllabus.

Problems:

1. **(36 points)** Answer the following questions, each of which is worth **6 points**. You must provide justification for your answer.

   (a) Is the angle between the vectors $\mathbf{u} = \langle 1, 2, 0, -4\rangle$ and $\mathbf{v} = \langle 0, 2, 1, 2\rangle$ obtuse?

   Solution. Yes, because $\mathbf{u} \cdot \mathbf{v} = -4 < 0$. □

   (b) Do the lines $\mathbf{r}(t) = \langle 1 + 2t, -t, 7 - 6t\rangle$ and $\mathbf{s}(t) = \langle 1 + 4t, 1 - 2t, 7 - 12t\rangle$ intersect?

   Solution. The lines are parallel since the directions vectors $\mathbf{u} = \langle 2, -1, -6\rangle$ and $\mathbf{v} = \langle 4, -2, -12\rangle$ are parallel. Therefore, either the lines are the same, or the lines do not intersect. A point on $\mathbf{r}(t)$ is $(1, 0, 7)$ but this point does not lie on $\mathbf{s}(t)$ because $\langle 1, 0, 7\rangle = \mathbf{s}(t)$ implies $t = -1$ and $t = 0$, a contradiction. So the lines do not intersect. □

   (c) Are the planes $2x + y - z = 2$ and $-6x - 3y + 9z = 0$ parallel?

   Solution. The normal vectors are $\langle 2, 1, -1\rangle$ and $\langle -6, -3, 9\rangle$ which are not parellel since $\langle 2, 1, -1\rangle \times \langle -6, -3, 9\rangle = \langle 6, -12, 0\rangle \neq \langle 0, 0, 0\rangle$. Two planes are parallel if and only if their normal vectors are parallel. Therefore, the planes are not parallel. □

   (d) Are the vectors $\mathbf{u} = \langle 1, 3, 2\rangle$, $\mathbf{v} = \langle 1, 0, 1\rangle$, and $\mathbf{w} = \langle 0, -3, -1\rangle$ coplanar?

   Solution. Compute the volume of the parallelepiped. From Weekly 2.1, this is given by the determinant

   $$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 3 & 2 \\ 0 & -3 & -1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ -3 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 0 & -3 \end{vmatrix} = (-3 - (-6)) + -3 = 0.$$

   Since the volume is 0, the vectors are coplanear. □

(e) Is the line $\mathbf{r}(t) = \langle 1-t, 2-t, 3+6t\rangle$ contained in the plane $6x+6y+2z = 24$?

Solution. A necessary condition is that the line is parallel to the plane, i.e., the direction of the line is perpendicular to the normal vector. The direction of the line is $\langle -1, -1, 6\rangle$ and the normal vector is $\langle 6, 6, 2\rangle$. Since their dot product is zero, they are perpendicular. Therefore, the line is contained in the plane if and only if the line and the plane intersect. The point $(1, 2, 3)$ is a point on the line (when $t = 0$) and clearly this point also satisfies the equation of the plane. Therefore, the line is contained in the plane. □

(f) Is the line $\mathbf{r}(t) = \langle 1+3t, 1+4t, t\rangle$ tangent to the curve $\mathbf{s}(t) = \langle t^2, t^3, \ln t\rangle$ at the point $(1, 1, 0)$?

Solution. When $t = 1$, we get $s(1) = \langle 1, 1, 0\rangle$. A necessary condition for the line to be tangent is that the line contains this point. It clearly does, since $r(0) = \langle 1, 1, 0\rangle$. Therefore, the line is tangent to the curve if and only if the direction of the line $\langle 3, 4, 1\rangle$ is parallel to the tangent vector $\mathbf{s}'(1) = \langle 2, 3, 1\rangle$. Clearly, this is not the case. So the line is not the tangent line to the curve at the given point. □

2. (12 points) Let $P = (1, 2, -3)$, $Q = (1, 2, 4)$, $R = (1, 0, 7)$, $S = (-1, -1, -2)$. Determine if the points P, Q, R and S are coplanar. That is, do P, Q, R and S lie in the same plane? You must justify your claim.

Solution. The points P, Q, R are not collinear and therefore uniquely determine a plane. Since they all have the same x-coordinate, it is clear that the scalar equation of this plane is $x = 1$. But S does not lie on this plane because the x coordinate of S is not equal to 1. Therefore, the four points are not coplanar. □

3. (15 points) Let p denote the plane defined by the scalar equation $3x + 2y - z = 1$ and let q denote the plane defined by $7x - y - 2z = 4$. Complete the following tasks, each of which is worth **5 points**.

(a) Show that the planes p and q intersect.

(b) Show that the planes p and q are not parallel.

(c) Find a vector-valued function $\mathbf{r}(t)$ that describes this line.

Solution. **(a)** Set $y = 0$. Then solve the system of equations $7x - 2z = 4$ and $3x - z = 1$ by substitution or elimination. You will get $x = 2$ and $z = 5$. Therefore, a point on both planes is $(2, 0, 5)$.

(b) The normal vectors are $\langle 3, 2, -1\rangle$ and $\langle 7, -1, -2\rangle$ which are not parallel because

$$\langle 3, 2, -1\rangle \times \langle 7, -1, -2\rangle = \langle -5, -1, -17\rangle$$

is not the zero vector.

(c) A point on the line is the point we found in (a), since it lies on both planes. The direction of the line must be perpendicular to the normal vectors of both planes. Therefore, the direction of the line is given by the cross product of the two normal vectors, which we already found in part (b):

$$\langle -5, -1, -17\rangle.$$

Therefore, a vector-valued functiond describing the line is given by $\mathbf{r}(t) = \langle 2, 0, 5\rangle + t\langle -5, -1, -17\rangle$.

□

4. (18 points) Let $f(x,y) = 3x^2 + 2y^2$. Complete each of the following tasks, each of which is worth **3 points**.

(a) Write down any nonzero point $P_0 = (x_0, y_0, z_0)$ that lies on the graph of the function $f(x,y)$.

(b) Find a vector-valued function $\mathbf{r}(t)$ that describes the $x = x_0$ trace of the function f.

(c) Find a vector-valued function $\mathbf{s}(t)$ that describes the $y = y_0$ trace of the function f.

(d) Find the direction $\mathbf{v}$ of the line containing P_0 whose direction is tangent to the $x = x_0$ trace at P_0. Compute the vector form $\mathbf{L_1}(t)$ of this line.

(e) Find the direction $\mathbf{w}$ of the line containing P_0 whose direction is tangent to the $y = y_0$ trace at P_0. Compute the vector form $\mathbf{L_2}(t)$ of this line.

(f) The *tangent plane* to the graph of f at P_0 is the plane containing both of the tangent lines $\mathbf{L_1}$ and $\mathbf{L_2}$ from part (d) and (e). Find the scalar equation of this plane.

Solution. **(a)** I chose $P_0 = (1, 0, 3)$.

(b) I need to parameterize the $x = 1$ trace. One possibility is given by $\mathbf{r}(t) = \langle 1, t, 3 + 2t^2\rangle$.

(c) I need to parameterize the $y = 0$ trace. One possibility is given by $\mathbf{s}(t) = \langle t, 0, 3t^2\rangle$.

(d) Notice that $\mathbf{r}(0) = \langle 1, 0, 3\rangle$. Therefore, the direction of the tangent line is given by $\mathbf{r}'(0) = \langle 0, 1, 0\rangle$. Then $\mathbf{L}_1(t) = \langle 1, 0, 3\rangle + t\langle 0, 1, 0\rangle$.

(e) Notice that $\mathbf{s}(1) = \langle 1, 0, 3\rangle$. Therefore, the direction of the tangent line is given by $\mathbf{s}'(1) = \langle 1, 0, 6\rangle$. Then $\mathbf{L}_2(t) = \langle 1, 0, 3\rangle + t\langle 1, 0, 6\rangle$.

(f) The tangent plane contains $(1, 0, 3)$ and has normal vector given by

$$\langle 0, 1, 0\rangle \times \langle 1, 0, 6\rangle = \langle 6, 0, -1\rangle.$$

Therefore, a scalar equation of the plane is given by

$$6(x - 1) - (z - 3) = 0$$

or

$$6x - z = 3.$$

□

5. (12 points) Let $f(x,y) = 3x^2 + 2y^2$ and let $P_0 = (x_0, y_0, z_0)$ be the point you chose in part (a) of the preceding problem. Complete the following tasks, each of which is worth **4 points**.

(a) Compute $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$.

(b) Explain the relationship between $f_y(x_0, y_0)$ and $\mathbf{L_1}$.

(c) Explain the relationship between $f_x(x_0, y_0)$ and $\mathbf{L_2}$.

Solution. (a) We see that $f_x(x,y) = 6x$ and $f_y(x,y) = 4y$ so that $f_x(1,0) = 6$ and $f_y(1,0) = 0$.

(b) The quantity $f_y(1,0) = 0$ is the slope of the line $\mathbf{L_1}$ when considered as a tangent line to the $x = 1$ trace of f.

(c) The quantity $f_x(1,0) = 6$ is the slope of the line $\mathbf{L_2}$ when considered as a tangent line to the $x = 1$ trace of f.

□

6. (12 points) The table below gives the number $C(W, S)$ of calories burned per hour for a person roller-blading as a function of the person's weight W in pounds and speed S in miles per hour.

W\S	8	9	10	11
120	252	348	444	534
140	306	402	498	594
160	366	462	552	648
180	420	516	612	702
200	474	570	666	756

Answer the following questions, each of which is worth **6 points**.

(a) Estimate the partial derivative $C_W(140, 10)$. Make sure to include the units.

(b) Estimate the partial derivative $C_S(140, 10)$. Make sure to include the units.

Solution. As seen in class, we can approximate partial derivatives using the symmetric difference $\frac{g(t+h)-g(t-h)}{2h}$ where g is a function whose graph is an appropriate trace of C.

(a) To approximate $C_W(140, 10)$, we need to hold S constant. The W values are in increments of 20, so we use $h = 20$. We have

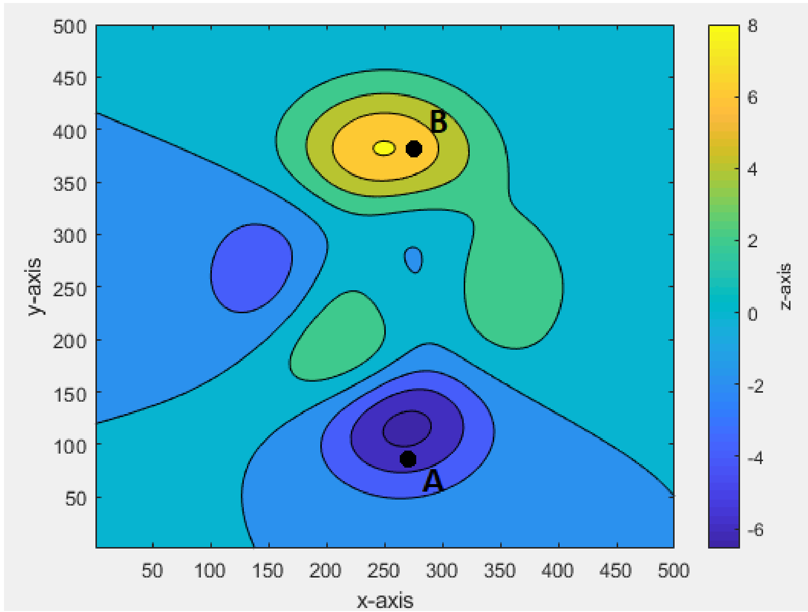
$$\frac{C(140+20, 10) - C(140-20, 10)}{2 \cdot 20} = \frac{552 - 444}{40} = \frac{108}{40} \text{cal/(lb} \cdot \text{hr)}.$$

(b) To approximate $C_S(140, 10)$, we need to hold W constant. The S values are in increments of 1, so we use $h = 1$. We have

$$\frac{C(140, 10+1) - C(140, 10-1)}{2 \cdot 1} = \frac{594 - 402}{2} = 96\text{cal/mi}.$$

□

7. (12 points) Consider the following contour diagram of a function $f(x, y)$:



I have identified two points A and B in the figure. Complete the following tasks, each of which is worth **3 points**.

(a) Determine whether f_x is positive, negative, or zero at point A.

(b) Determine whether f_y is positive, negative, or zero at point A.

(c) Determine whether f_x is positive, negative, or zero at point B.

(d) Determine whether f_y is positive, negative, or zero at point B.

Solution. **(a)** Recall that f_x is the slope of the tangent line to a trace in the x direction. At the point A, the slope is zero according the contour map.

(b) Recall that f_y is the slope of the tangent line to a trace in the y direction. At the point A, the slope is positive.

(c) At the point B, the slope of the tangent line in the x direction is negative. So f_x is negative.

(d) At the point B, the slope of the tangent line in the y direction is zero, so f_y is zero.

□