

Name: _____

15 minutes

This quiz has 2 pages consisting of 3 problems worth a total of 13 points, plus a 1-point bonus question. *However, the quiz is graded out of 12 points.* You may invoke any theorem that was proved in class, provided you make it clear which result you are using. For example, you might say something like “since $a \mid b$, we have $a \mid bc$,” or “by the transitive property of congruence modulo n .” You may use reasonable shorthand in your proofs, such as writing “I.H.” for the inductive hypothesis, or logical symbols like $\Rightarrow$, $\forall$, and $\exists$, provided that your reasoning is clear.

Problem 1. (5 points, 1 point each) Determine whether each statement is true or false. Write **TRUE** or **FALSE** in the box; no justification is required.

- (a) Let $a, b \in \mathbb{N}$ with $a > b$. If the Euclidean Algorithm applied to a and b halts in k steps, then the Euclidean Algorithm applied to a and $a + b$ halts in $k + 2$ steps. Answer:
- (b) Let $a, b \in \mathbb{Z}$. If there exist $x, y \in \mathbb{Z}$ such that $ax + by = 6$, then $(a, b) = 6$. Answer:
- (c) Let $a, b, c \in \mathbb{Z}$. If $a \mid bc$ and $a \nmid b$, then $a \mid c$. Answer:
- (d) Let $a, b, n \in \mathbb{Z}$ with $n > 0$. If $a \equiv b \pmod{n}$, then $(a, n) = (b, n)$. Answer:
- (e) Let $a, b, q, q', r, r' \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $a = qn + r$ and $b = q'n + r'$ with $r \neq r'$, then a and b are not congruent modulo n . Answer:

Problem 2. (4 points) Let $a, b \in \mathbb{Z}$, not both 0, and let $n \in \mathbb{N}$. Prove that $(na, nb) = n(a, b)$.

Problem 3. (4 points) The **Fibonacci numbers** are defined by $F_1 = 1$, $F_2 = 1$, and

$$F_{n+2} = F_{n+1} + F_n \quad \text{for } n \geq 1.$$

Prove, using the Principle of Mathematical Induction, that F_n and F_{n+1} are relatively prime for every $n \geq 1$. You may use without proof that $F_n \geq 1$ for every $n \geq 1$.

Bonus question (1 point). Let $a, b \in \mathbb{Z}$, not both 0, and define $L_1 = a$, $L_2 = b$, and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 1$. Generalize Problem 3 by stating a theorem that determines (L_n, L_{n+1}) . Your theorem must be a complete and precise mathematical sentence, including all hypotheses and quantifiers. You need not prove your claim.