

Motion in Space

Name: _____

Stewart §13.2 and §13.4.

THE DERIVATIVE OF A VECTOR-VALUED FUNCTION

The derivative of $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is defined exactly as for real-valued functions:

$$\mathbf{r}'(t) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h},$$

In practice, it is computed component-wise: $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$. Let P be the point with position vector $\mathbf{r}(t)$. If $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq \mathbf{0}$, the vector $\mathbf{r}'(t)$ is called the **tangent vector** to the curve at P . The **tangent line** to C at P is the line through P parallel to $\mathbf{r}'(t)$.

WHY $\mathbf{r}'(t) = \mathbf{0}$

DIFFERENTIATION RULES

The derivative satisfies many of the familiar properties from Calculus I. Below, $\mathbf{r}$ and $\mathbf{s}$ are differentiable vector functions, c is a constant, and f is a differentiable scalar function.

$$\frac{d}{dt} [\mathbf{r}(t) + \mathbf{s}(t)] = \underline{\hspace{2cm}} \qquad \frac{d}{dt} [\mathbf{r}(t) \cdot \mathbf{s}(t)] = \underline{\hspace{2cm}}$$

$$\frac{d}{dt} [c \mathbf{r}(t)] = \underline{\hspace{2cm}} \qquad \frac{d}{dt} [\mathbf{r}(t) \times \mathbf{s}(t)] = \underline{\hspace{2cm}}$$

$$\frac{d}{dt} [f(t) \mathbf{r}(t)] = \underline{\hspace{2cm}} \qquad \frac{d}{dt} [\mathbf{r}(f(t))] = \underline{\hspace{2cm}}$$

CONSTANT MAGNITUDE

UNIT TANGENT, UNIT NORMAL, BINORMAL

ANTIDERIVATIVES

An **antiderivative** of $\mathbf{r}$ is a vector function $\mathbf{R}$ with $\mathbf{R}' = \mathbf{r}$, and an **indefinite integral** of $\mathbf{r}$ is any antiderivative of $\mathbf{r}$. Differentiation happens componentwise, so

$$\int \mathbf{r}(t) dt = \left\langle \int f(t) dt, \int g(t) dt, \int h(t) dt \right\rangle.$$

Any two antiderivatives differ by a constant, so

$$\int \mathbf{r}(t) dt = \mathbf{R}(t) + \mathbf{C}$$

where $\mathbf{C}$ is some constant vector.

Activity 1 _____ Integrating componentwise

With the person next to you.

(a) Compute $\int \left\langle \sin t, \frac{1}{1+t^2}, 2t \right\rangle dt$.

(b) Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = \langle 2t, \sin t \rangle$ and $\mathbf{r}(0) = \langle 1, 0 \rangle$.

THE DEFINITE INTEGRAL

If $\mathbf{r}$ is continuous, the definite integral $\int_a^b \mathbf{r}(t) dt$ can be defined as a limit of Riemann sums. Everything again happens componentwise, and the Fundamental Theorem of Calculus extends verbatim:

where $\mathbf{R}$ is _____.

(c) Compute $\int_0^1 \langle 2t, 3t^2, e^t \rangle dt$.

VELOCITY, SPEED, AND ACCELERATION

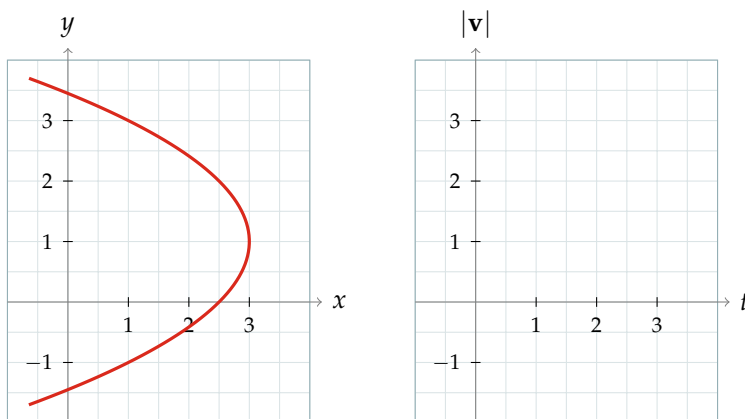
Let $\mathbf{r}(t)$ be the position of a particle at time t . Then

$$\text{velocity} = \mathbf{v}(t) = \mathbf{r}'(t), \quad \text{speed} = |\mathbf{v}(t)|, \quad \text{acceleration} = \mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t).$$

Activity 2 _____ **Velocity, speed, and acceleration¹**

With the person next to you. The left side of the figure below shows the curve described by the vector-valued function $\mathbf{r}$ defined by

$$\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1 \rangle.$$



(a) Find the object's velocity $\mathbf{v}(t)$ and acceleration $\mathbf{a}(t)$.

¹Activity 9.7.4 of *Active Calculus: Multivariable*, part (g) omitted. CC BY-SA 4.0.

- (b) On the left, mark the object's position at $t = 0, 2, 4$. Draw the velocity and acceleration vectors with their tails placed at the object's position.
- (c) Find the object's speed $|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$ and graph it on the right above. When is it slowest? Increasing? Decreasing?
- (d) From your drawing, what seems to be true about the angle between $\mathbf{v}$ and $\mathbf{a}$ at the minimum? When the speed is increasing? Decreasing?
- (e) The speed increases exactly when $\mathbf{v} \cdot \mathbf{v}$ does. Use the product rule to express $\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v})$ in terms of $\mathbf{v}$ and $\mathbf{a}$, and compare with (d).

SPEEDING UP AND SLOWING DOWN



Activity 3 Going backwards

At the board with your group. *I will assign the groups.* Roles by **first** name in **alphabetical** order: first is **scribe**, second is **verifier**, third is **reporter**.

GROUP WORK — AT THE BOARD

Find your group number below. You are given the acceleration of a particle together with its velocity and position at one instant.

Groups 1, 5 $\mathbf{a}(t) = \langle 6t, e^t, 0 \rangle, \quad \mathbf{v}(0) = \langle 1, 0, 2 \rangle, \quad \mathbf{r}(0) = \langle 1, 0, 0 \rangle$

Groups 2, 6 $\mathbf{a}(t) = \langle \sin t, 6t, 0 \rangle, \quad \mathbf{v}(0) = \langle 1, 0, 2 \rangle, \quad \mathbf{r}(0) = \langle 0, 1, 0 \rangle$

Groups 3, 7 $\mathbf{a}(t) = \langle 2, \cos t, 6t \rangle, \quad \mathbf{v}(0) = \langle 0, 2, 1 \rangle, \quad \mathbf{r}(0) = \langle 1, 0, 0 \rangle$

Groups 4, 8 $\mathbf{a}(t) = \langle 0, 6t, e^t \rangle, \quad \mathbf{v}(0) = \langle 2, 1, 0 \rangle, \quad \mathbf{r}(0) = \langle 0, 0, 2 \rangle$

On the board:

1. Find $\mathbf{v}(t)$ and $\mathbf{r}(t)$.
2. Check your answer by differentiating $\mathbf{r}$ twice.

Hint: integrate to find $\mathbf{v}$; use the initial condition to find the constant of integration. Repeat to get $\mathbf{r}$.

Done early? At time $t = 1$ the particle flies off the path and keeps going in a straight line at constant velocity. Where is it at time t , for $t \geq 1$? Parametrize so that the clock keeps running.

Discussion. Two particles have the same acceleration at every time t . How different can their paths be? What if they also have the same velocity at $t = 0$?

PROJECTILE MOTION

An object is launched from the origin with initial speed v_0 at an angle θ above the horizontal, so that $\mathbf{v}(0) = \langle v_0 \cos \theta, v_0 \sin \theta \rangle$. If gravity is the only force acting on it, then $\mathbf{a}(t) = \langle 0, -g \rangle$.

Activity 4 _____ **Projectile motion**

At the board with your same group. **Rotate roles:** the **verifier** is now scribe, the **reporter** is now verifier, and the **scribe** is now reporter.

GROUP WORK — AT THE BOARD

Steps 1 and 2 are the same for everyone. Use $v_0 = 20$ m/s and $g = 9.8$ m/s², and your group's angle below.

Groups 1, 5

$\theta = 30^\circ$

Groups 2, 6

$\theta = 45^\circ$

Groups 3, 7

$\theta = 60^\circ$

Groups 4, 8

$\theta = 75^\circ$

On the board, large enough to read from across the room:

1. Find $\mathbf{v}(t)$ and $\mathbf{r}(t)$ in terms of v_0 , θ , and g .
2. When does the object land? (Solve for the time the height returns to zero.)
3. How far away does it land? This is the **range**.
4. How high does it get?

Done early? Which launch angle gives the greatest range? Prove it.

Discussion. Two groups got the same range. Which two, and why? Did anything else about their flights agree?