

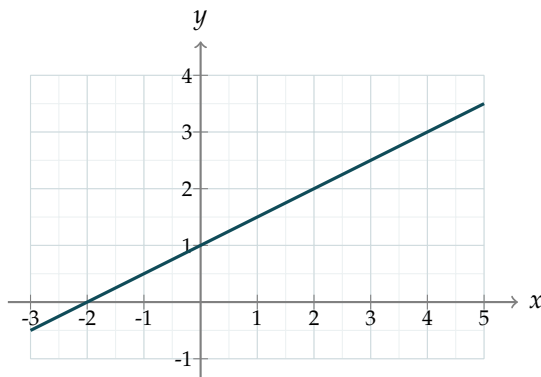
Equations of Lines and Planes

Name: _____

Stewart §12.5.

Warm-up: describe this line with vectors

In Activity 2 from Worksheet 3, we discovered that the set of terminal points of the vectors $\mathbf{u} + t\mathbf{v}$, as t ranges over $\mathbb{R}$, is the line through the tip of $\mathbf{u}$ in the direction of $\mathbf{v}$. We can think of this as a **vector-valued function** $\mathbf{r}(t) = \mathbf{u} + t\mathbf{v}$ which takes in a real number t and returns a vector. The components of $\mathbf{r}(t)$ are ordinary functions $x(t)$ and $y(t)$.



With the person next to you. Suppose you are given the line $y = \frac{1}{2}x + 1$ which is graphed above.

- Plot the points $P = (0, 1)$ and $Q = (2, 2)$ on the line. Draw the **position vectors** $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ from the origin O .
- Find a vector $\mathbf{v}$ parallel to the line. Draw $\mathbf{v}$ on the graph with its tail at P . How are its components related to the slope?
- Write a vector equation $\mathbf{r}(t)$ for this line. Then find the value of t for which $\mathbf{r}(t)$ has its tip at $R = (4, 3)$, and draw that vector on the graph.
- Write the components $x(t)$ and $y(t)$ of $\mathbf{r}(t)$.

VECTOR AND PARAMETRIC EQUATIONS OF A LINE

A **vector equation** for a line L in $\mathbb{R}^3$ is given by

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \quad t \in \mathbb{R},$$

where $\mathbf{r}$ is the position vector of a point on L , $P_0 = (x_0, y_0, z_0)$ is a point on L , $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ is the position vector for P_0 , and $\mathbf{v} = \langle a, b, c \rangle$ is a vector parallel to L . The **parametric equations** of L are given by

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

Activity 1 _____ **Writing equations of lines**

With the person next to you.

- (a) Give a vector equation and parametric equations for the line L_1 through $(1, 2, 3)$ with direction $\langle 1, 2, 2 \rangle$.

- (b) Give a vector equation and parametric equations for the line L_2 through $P = (2, 4, 5)$ and $Q = (4, 8, 9)$.

- (c) A third line L_3 is given by $x = 3 + 3s$, $y = 6 + 6s$, $z = 8 + 6s$. Write down a point on L_3 and a direction vector for it. Then decide: are any two of L_1 , L_2 , L_3 the same line? Are any of them parallel? How do you know?

Activity 2 _____ Getting rid of t

In the warm-up you started from an equation in x and y and produced a parameterization. Now go the other way.

- (a) Take your parametric equations from Activity 1(a) and solve each one for t . Setting the three results equal to one another gives the **symmetric equations** of L_1 . Write them down.

- (b) That chain contains three equations: one in x and y , one in y and z , and one in x and z . Write all three down and explain why one of the equations is redundant.

- (c) Solve each equation in (b) for one of the variables. What does each equation look like in $\mathbb{R}^2$? What does each equation look like in $\mathbb{R}^3$? What about two of them together?

- (d) Use your symmetric equations to decide whether $(6, 12, 13)$ lies on L_1 . What about $(6, 12, 14)$? Explain why this is quicker than using the parametric equations.

- (e) Repeat part (a) for the line through $(1, 2, 3)$ with direction $\langle 1, 2, 0 \rangle$. What goes wrong, and what does the line look like?

SYMMETRIC EQUATIONS OF A LINE

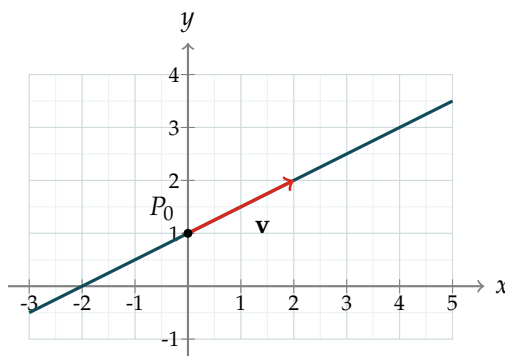
The **symmetric equations** of the line through $P_0 = (x_0, y_0, z_0)$ parallel to $\mathbf{v} = \langle a, b, c \rangle$ are given by

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c},$$

provided a , b , and c are all nonzero.

Activity 3 _____ Lines, using the dot product

We are going to consider the line from the warm-up activity again. This is the line through the point $P_0 = (0, 1)$ and parallel to $\mathbf{v} = \langle 2, 1 \rangle$. Let $P = (x, y)$ be an arbitrary point of $\mathbb{R}^2$, and write $\mathbf{r}_0 = \langle 0, 1 \rangle$ and $\mathbf{r} = \langle x, y \rangle$ for the position vectors of P_0 and P .



- Find a nonzero vector $\mathbf{n}$ orthogonal to $\mathbf{v}$, that is, with $\mathbf{n} \cdot \mathbf{v} = 0$. Draw it on the graph above with its tail at P_0 .
- Mark a point P of your choosing on the line. Draw $\mathbf{r}_0$, $\mathbf{r}$, and $\mathbf{r} - \mathbf{r}_0$ on the graph. How is $\mathbf{r} - \mathbf{r}_0$ related to $\mathbf{v}$? What changes if P is not on the line?
- Combine (a) and (b): when P is on the line, how is $\mathbf{n}$ related to $\mathbf{r} - \mathbf{r}_0$? Write down an equation using the dot product that describes this relationship. Explain why the equation fails when P is not on the line.
- Expand the dot product to obtain an equation in terms of x and y . Where have you seen this equation before?

NORMAL VECTORS AND THE SCALAR EQUATION OF A LINE

A line in $\mathbb{R}^2$ is determined by a point $P_0 = (x_0, y_0)$ on the line together with a **normal vector** denoted $\mathbf{n}$ which is orthogonal to every vector parallel to the line. If $P = (x, y)$ is any point, with position vectors $\mathbf{r}$ and $\mathbf{r}_0$ for P and P_0 , then P is on the line exactly when

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0.$$

Writing $\mathbf{n} = \langle a, b \rangle$ gives the **scalar equation** of the line,

$$a(x - x_0) + b(y - y_0) = 0.$$

THE SAME EQUATION IN $\mathbb{R}^3$

Let $\mathbf{n}$ be a nonzero vector in $\mathbb{R}^3$, and let P_0 be a point with position vector $\mathbf{r}_0$. What does the equation

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

describe in $\mathbb{R}^3$?