

The Cross Product

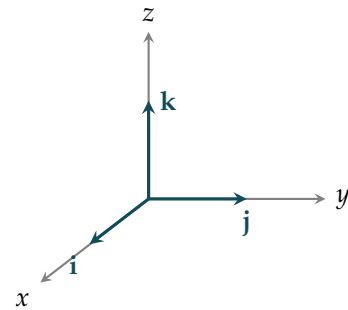
Name: _____

Stewart §12.4.

Debrief of Preview Activity 3:

In the Preview Activity, you computed the cross product on pairs of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ using the right hand rule.

$\times$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$			
$\mathbf{j}$			
$\mathbf{k}$			

Table 1. The entry in row $\mathbf{u}$ and column $\mathbf{v}$ is $\mathbf{u} \times \mathbf{v}$.Figure 1. Basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

The **cross product** is the binary operation $\times$ on vectors in $\mathbb{R}^3$ that agrees with Table 1 and is *linear in each argument*:

$$\begin{aligned}(c\mathbf{u}) \times \mathbf{v} &= c(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (c\mathbf{v}) \\ (\mathbf{u} + \mathbf{v}) \times \mathbf{w} &= (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w}) \\ \mathbf{u} \times (\mathbf{v} + \mathbf{w}) &= (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})\end{aligned}$$

Formula for $\mathbf{u} \times \mathbf{v}$:

THE COMPONENT FORMULA

DETERMINANTS

DETERMINANT FORMULA FOR THE CROSS PRODUCT

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

The last matrix has vectors in the first row, so it's "determinant" is a vector, not a number.

Activity 1 _____ Computing cross products¹

With the person next to you. Let $\mathbf{u} = \langle 0, 1, 3 \rangle$ and $\mathbf{v} = \langle 2, -1, 0 \rangle$. Use the determinant formula for the following.

- (a) Find the cross product $\mathbf{u} \times \mathbf{v}$.

- (b) Evaluate the dot products $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$ and $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})$. What does this tell you about the geometric relationship among $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$?

- (c) Find the cross product $\mathbf{v} \times \mathbf{i}$.

- (d) Multiplication of real numbers is *associative*, which means, for instance, that $(2 \cdot 5) \cdot 3 = 2 \cdot (5 \cdot 3)$. Is it true that the cross product of vectors is associative? For instance, is it true that $(\mathbf{u} \times \mathbf{v}) \times \mathbf{i} = \mathbf{u} \times (\mathbf{v} \times \mathbf{i})$?

- (e) Find the cross product $\mathbf{u} \times \mathbf{u}$.

¹Activity 9.4.2 of *Active Calculus — Multivariable* (Schlicker, Keller, Long), activecalculus.org, used under CC BY-SA 4.0.

PROPERTIES OF THE CROSS PRODUCT

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors in $\mathbb{R}^3$ and let c be a scalar. Then

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
2. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$
3. $(c\mathbf{u}) \times \mathbf{v} = c(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (c\mathbf{v})$
4. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel
5. the cross product is not associative

The length of $\mathbf{u} \times \mathbf{v}$:

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$, and let $0 \leq \theta \leq \pi$ be the angle between them. Then

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin \theta.$$

You checked this for two vectors in the xy -plane in part (g) of the Preview Activity.

GEOMETRIC DESCRIPTION OF $|\mathbf{u} \times \mathbf{v}|$

Activity 2 _____ Areas²

With the person next to you.

(a) Find the area of the parallelogram formed by the vectors $\mathbf{u} = \langle 1, 3, -2 \rangle$ and $\mathbf{v} = \langle 3, 0, 1 \rangle$.

(b) Find the area of the parallelogram in $\mathbb{R}^3$ whose vertices are $(1, 0, 1)$, $(0, 0, 1)$, $(2, 1, 0)$, and $(1, 1, 0)$.
(Hint: it might help to draw a picture, to see how the vertices are arranged and which vectors you might use.)

The direction of $\mathbf{u} \times \mathbf{v}$:

GEOMETRIC DESCRIPTION OF DIRECTION OF $\mathbf{u} \times \mathbf{v}$

²Activity 9.4.3 of the same source.

GEOMETRIC DEFINITION OF $\mathbf{u} \times \mathbf{v}$

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$. Then $\mathbf{u} \times \mathbf{v}$ is the unique vector satisfying:

1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, and $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} \times \mathbf{v}$ satisfy the right-hand rule;
2. $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$, the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$, where $0 \leq \theta \leq \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

The scalar triple product:**THE SCALAR TRIPLE PRODUCT****Example** _____ **Coplanar vectors³**

Do the vectors $\mathbf{a} = \langle 1, 3, -2 \rangle$, $\mathbf{b} = \langle 2, 1, -4 \rangle$, and $\mathbf{c} = \langle 0, 1, 0 \rangle$ in standard position lie in the same plane? Use the concepts from this section to explain.

³Part (e) of Activity 9.4.4 of the same source.