The Dot Product

Name: ____________________

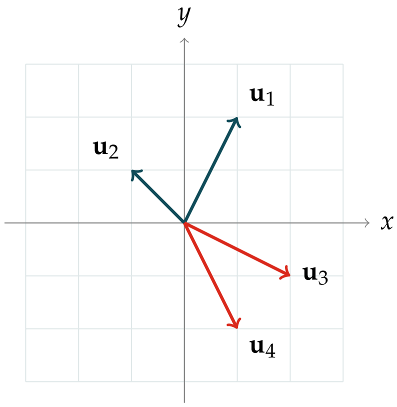
Stewart §12.3. Today's question: what is the relationship between the dot product and the angle between two vectors?

From Preview Activity 2: The dot product of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ in $\mathbb{R}^2$ is defined to be $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$. The same definition makes sense in any dimension $n \geq 1$: the **dot product** of $\mathbf{u} = \langle u_1, \ldots, u_n \rangle$ and $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$ in $\mathbb{R}^n$ is the *scalar*

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

In Preview Activity 2 you discovered that this number contains information about the geometric relationship between $\mathbf{u}$ and $\mathbf{v}$.

Debrief of Preview Activity 2:



Dot product	Value	θ
$\mathbf{u}_1 \cdot \mathbf{u}_2$	1	acute
$\mathbf{u}_1 \cdot \mathbf{u}_3$	0	right
$\mathbf{u}_1 \cdot \mathbf{u}_4$	-3	obtuse
$\mathbf{u}_2 \cdot \mathbf{u}_4$	-3	obtuse
$\mathbf{u}_3 \cdot \mathbf{u}_4$	4	acute

Observations:

Properties of the dot product. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and let c be a scalar. Then

1. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$

2. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

4. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

5. $\mathbf{0} \cdot \mathbf{u} = 0$

The dot product properties are all inherited from addition and multiplication in $\mathbb{R}$.

Activity 1 — Using dot product properties[1]

With the person next to you. Use the dot product properties to answer the following questions.

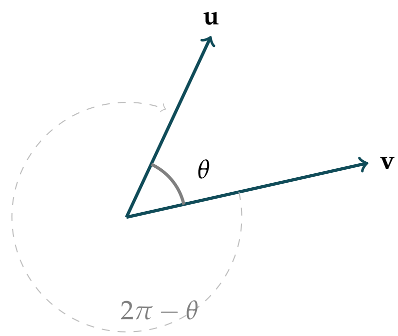
(a) Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^n$ with $\mathbf{u} \cdot \mathbf{w} = 10$ and $\mathbf{v} \cdot \mathbf{w} = -3$. Compute $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w}$. Which property are you using?

(b) Let $\mathbf{u}, \mathbf{v}$ be vectors in $\mathbb{R}^n$ with $\mathbf{u} \cdot \mathbf{v} = 3$, $|\mathbf{u}| = 4$, and $|\mathbf{v}| = 7$. Compute $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})$. What is $|\mathbf{u} + \mathbf{v}|$?

(c) Let $\mathbf{u}, \mathbf{v}$ be vectors in $\mathbb{R}^n$ with $\mathbf{v} \neq \mathbf{0}$. For what value of t is $(t\mathbf{u} + \mathbf{v}) \cdot \mathbf{v} = 0$?

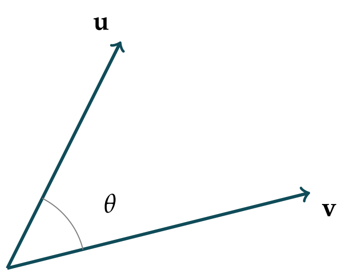
[1] Adapted from Activity 9.4.2 of *Active Calculus — Multivariable*, 2nd ed., by Nicholas Long and Mitchel T. Keller, activecalculus.org, used under CC BY-SA 4.0.

The angle between two vectors:



Two vectors **u** and **v** drawn from the same initial point make two angles. The **angle** θ **between u and v** is defined to be the smaller of the two. By definition, this angle satisfies $0 \leq \theta \leq \pi$.

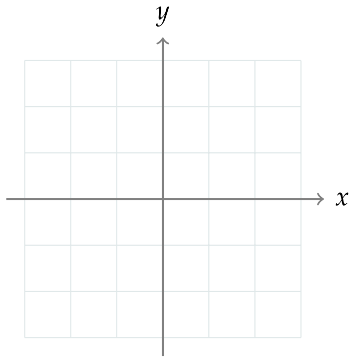
The angle formula



Activity 2 — Measuring angles[2]

With the person next to you. **No calculators.** Every angle below can be calculated using the unit circle.

(a) Find the angle between $\langle 2, 2\rangle$ and $\langle 0, 3\rangle$. Then find the angle between $\langle 2, 2\rangle$ and $\langle -3, 0\rangle$. Sketch both pairs. What do you notice about the two answers?



(b) Find the angle between $\langle 1, 1, 0\rangle$ and $\langle 0, 1, 1\rangle$. Before you compute: what do you expect?

ANGLES & ORTHOGONALITY

[2] Adapted from Activity 9.4.3 of the same source.

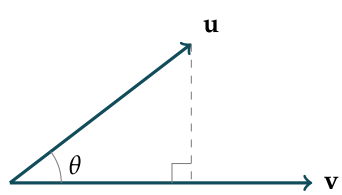
Direction cosines

Orthogonal projections:

Recall:

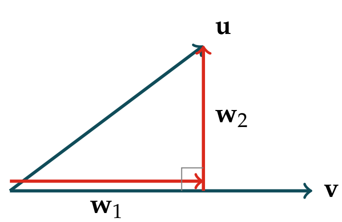
$$\mathbf{u} \cdot \mathbf{v} \;=\; |\mathbf{v}| \left(|\mathbf{u}| \cos\theta \right).$$

What does $|\mathbf{u}|\cos(\theta)$ mean geometrically?



Activity 3 — Splitting a vector[3]

With the person next to you. We want to write any vector $\mathbf{u}$ as a sum of two vectors $\mathbf{w}_1$ and $\mathbf{w}_2$, where $\mathbf{w}_1$ is parallel to $\mathbf{v}$ and $\mathbf{w}_2$ is orthogonal to $\mathbf{v}$.



(a) Using the picture, explain why $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$.

[3] Adapted from Activity 9.4.6 of the same source.

(b) Compute $\mathbf{u} \cdot \mathbf{v}$ by writing it as $(\mathbf{w}_1 + \mathbf{w}_2) \cdot \mathbf{v}$ and simplifying as far as you can. Use the fact that $\mathbf{w}_2$ is orthogonal to $\mathbf{v}$.

(c) Since $\mathbf{w}_1$ is parallel to $\mathbf{v}$, there is a scalar k with $\mathbf{w}_1 = k\mathbf{v}$. Substitute and solve for k.

(d) Write $\mathbf{w}_1$ using only $\mathbf{u}$ and $\mathbf{v}$. Then do the same for $\mathbf{w}_2$.

(e) Let $\mathbf{u} = \langle 2, 6 \rangle$ and $\mathbf{v} = \langle 3, 3 \rangle$. Find $\mathbf{w}_1$ and $\mathbf{w}_2$, and check both that $\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{u}$ and that $\mathbf{w}_2 \cdot \mathbf{v} = 0$. Then: what would change if $\mathbf{v}$ were $\langle -3, -3 \rangle$ instead?

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