## Vectors

Name: ______________________

Stewart §12.2. Today's question: how do we visualize vectors and their operations?

**From Preview Activity 1 and the reading:**
A **point** $(x,y)$ or $(x,y,z)$ is a location in space. A **vector** $\langle x,y\rangle$ or $\langle x,y,z\rangle$, which can be visualized as an arrow, records a *direction* and a *distance*. The **displacement** vector from $A=(x_1,y_1)$ to $B=(x_2,y_2)$ is

$$\overrightarrow{AB} = \langle x_2 - x_1,\ y_2 - y_1\rangle.$$

Vectors can be added componentwise

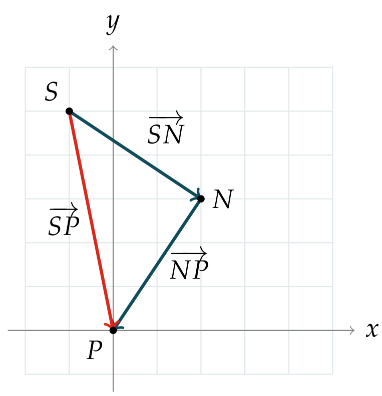
$$\langle u_1,u_2\rangle + \langle v_1,v_2\rangle = \langle u_1+v_1,\ u_2+v_2\rangle,$$

and scaled by a constant $c$

$$c\langle v_1,v_2\rangle = \langle cv_1,\ cv_2\rangle.$$

The **magnitude** (length ) of $\mathbf{v}=\langle v_1,v_2\rangle$ is $|\mathbf{v}| = \sqrt{v_1^2+v_2^2}$. A **unit vector** is a vector with $|\mathbf{v}|=1$.

## Debrief of Preview Activity 1:

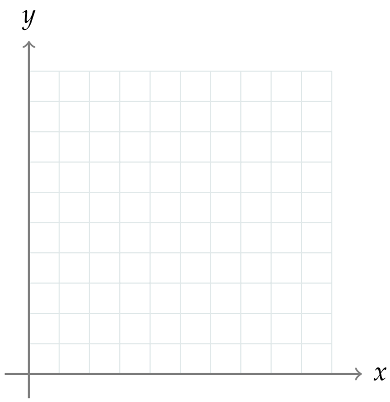



In the Preview Activity you discovered that $\overrightarrow{SN}+\overrightarrow{NP}=\overrightarrow{SP}$ algebraically by comparing components. Drawn as arrows, the tail of $\overrightarrow{NP}$ sits at the tip of $\overrightarrow{SN}$, and the sum runs from the tail of the first arrow to the tip of the second.

### Geometric vectors

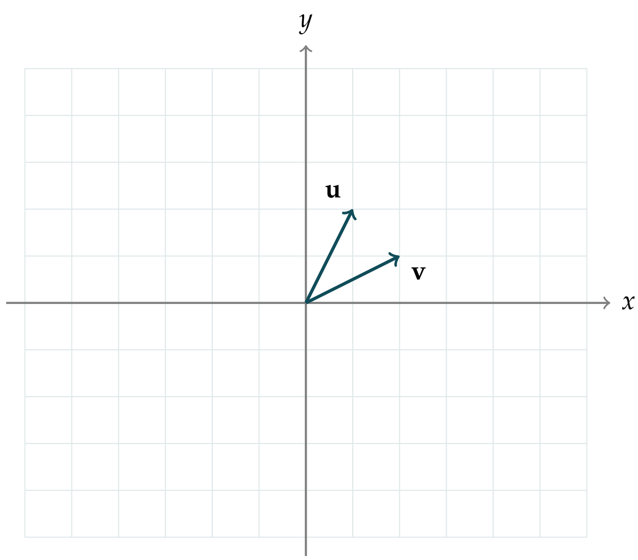
 

## Geometry of addition




### Activity 1 — Drawing multiples, sums, and differences[1]

With the person next to you. Let $\mathbf{u} = \langle 1, 2\rangle$ and $\mathbf{v} = \langle 2, 1\rangle$, drawn on the axes below.




(a) We define $-\mathbf{u} = (-1)\mathbf{u}$. Compute the components of $2\mathbf{u}$, $-\mathbf{u}$, and $\frac{1}{2}\mathbf{v}$, then sketch all three above. What does multiplying by $c$ do to the arrow? What does a negative $c$ do?

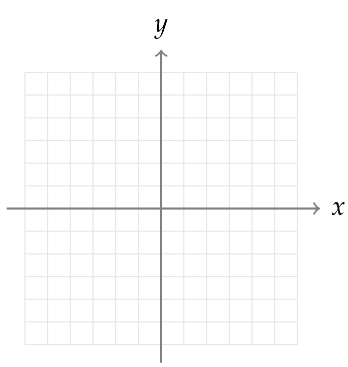
[1]Adapted from Activity 9.3.5 of *Active Calculus — Multivariable*, 2nd ed., by Nicholas Long and Mitchel T. Keller, activecalculus.org, used under CC BY-SA 4.0.

(b) Draw $\mathbf{u}+\mathbf{v}$ using the tip-to-tail method. Then compute it componentwise and check that the two methods agree. Repeat with $\mathbf{v}+\mathbf{u}$. What do you notice?

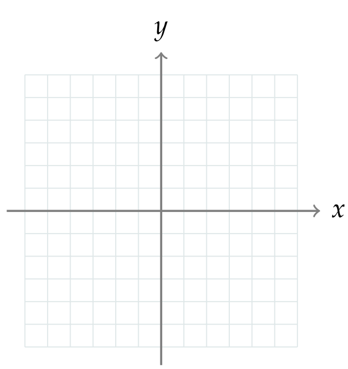
(c) We *define* subtraction of vectors as follows: $\mathbf{u}-\mathbf{v}=\mathbf{u}+(-\mathbf{v})$. Compute $\mathbf{u}-\mathbf{v}$ using the tip-to-tail method and sketch it. Then compute it componentwise and check that the two methods agree.

(d) Draw the arrow that starts at the tip of $\mathbf{v}$ and ends at the tip of $\mathbf{u}$. What are its components? Have you seen this vector before?

## Geometry of scalar multiplication




## Geometry of subtraction



## Activity 2 — What do the multiples of a vector trace out?

At the board with your group. *I will assign the groups.* Roles by **last** name in alphabetical order: first is **scribe**, second is **verifier**, third is **reporter**.

### GROUP WORK — AT THE BOARD

FIND YOUR GROUP BELOW.

| | |
|---|---|
| **Groups 1, 4** | sketch $t\mathbf{u}$ |
| **Groups 2, 5** | sketch $t\mathbf{v}$ |
| **Groups 3, 6** | sketch $\mathbf{u} + t\mathbf{v}$ |
| **Groups 4, 8** | sketch $\mathbf{v} + t\mathbf{u}$ |

(a) Draw a coordinate grid, running at least 6 units in each direction from the original. Draw $\mathbf{u}$ and $\mathbf{v}$.

(b) Compute your vectors algebraically for $t = -3, -2, -1, 0, 1, 2$.

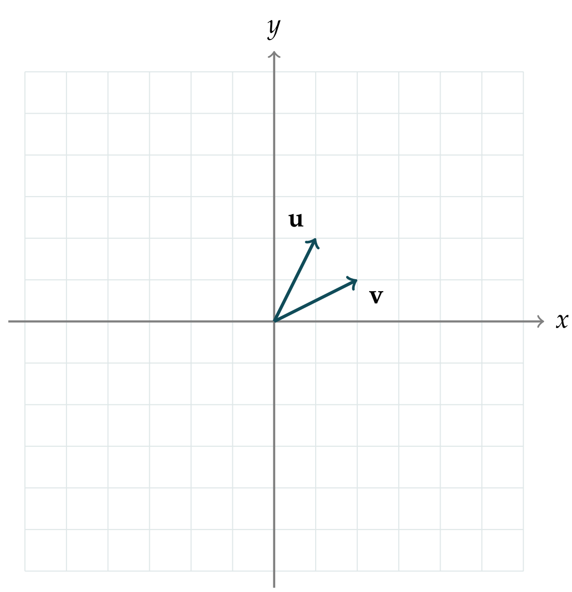
(c) Sketch them all on the grid. Mark the tip of each vector with a point and label it with the corresponding value of $t$.

(d) Based on your observations, describe in a sentence the set of *all* tips, as $t$ runs over every real number.

(e) Is every point of that set a tip for some $t$? How do you know?

(f) Find a parameterization for the curve you drew in part (d).

**Box your answers to (c) and (e).**




Fill in the rest of the table during the discussion.

| Groups | Vector | Curve | Parameterization |
|---|---|---|---|
| 1, 4 | $t\mathbf{u}$ | | |
| 2, 5 | $t\mathbf{v}$ | | |
| 3, 6 | $\mathbf{u} + t\mathbf{v}$ | | |
| 4, 8 | $\mathbf{v} + t\mathbf{u}$ | | |

What do all four have in common? What decides the direction of the line, and what decides where it starts?

**MULTIPLES, DIFFERENCES, AND LINES**

## Activity 3 — Why was the detour longer?[2]

Complete with the person next to you. Let $\mathbf{u} = \langle 2, 3\rangle$ and $\mathbf{v} = \langle -1, 2\rangle$. Recall that the **standard unit vectors** are $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$ in $\mathbb{R}^2$, and $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, $\mathbf{k} = \langle 0, 0, 1\rangle$ in $\mathbb{R}^3$.

(a) Find $|\mathbf{u}|$, $|\mathbf{v}|$, and $|\mathbf{u}+\mathbf{v}|$. Is $|\mathbf{u}+\mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$?

(b) Under what conditions does $|\mathbf{w}_1 + \mathbf{w}_2| = |\mathbf{w}_1| + |\mathbf{w}_2|$ hold? *Hint: $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_1 + \mathbf{w}_2$ are the three sides of a triangle.*

[2] Adapted from Activity 9.3.6 of the same source.

(c) In the Preview Activity the sum of your friends two vectors was equal to yours and yet your friend walked farther than you did. Explain how both of these can be true at once.

(d) With $\mathbf{u} = \langle 2, 3 \rangle$, find the lengths of $2\mathbf{u}$, $3\mathbf{u}$, and $-2\mathbf{u}$. Label each result with proper notation.

(e) In general, if $t$ is any scalar, how is $|t\mathbf{w}|$ related to $|\mathbf{w}|$?

(f) Of the vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, and $\mathbf{i} + \mathbf{j} + \mathbf{k}$, which are unit vectors?

(g) Find a unit vector with the same direction as $\mathbf{u} = \langle -2, 3 \rangle$. Then find a unit vector in the opposite direction of $\mathbf{u}$.

**Unit vectors**