

Constrained Optimization (Lagrange Multipliers)

Name: _____

Stewart §14.8.

Constrained optimization _____ Optimizing along a curve

In order to find the extreme values of a function over a closed and bounded region, we have to solve a *constrained optimization problem* on the boundary of the region. The method of Lagrange multipliers solves a problem like this without parametrizing the constraint curve.

THE CONSTRAINED OPTIMIZATION PROBLEM

A **constrained optimization problem** asks for the absolute maximum and minimum values of a function $f(x, y)$ subject to a *constraint equation* $g(x, y) = k$, where k is a constant. That is, we want to find the absolute maximum and minimum values of $f(x, y)$ among all points (x, y) on the level curve $g(x, y) = k$.

WHERE CAN THE EXTREME VALUES OCCUR?

LAGRANGE'S THEOREM

Suppose f and g are differentiable. If f attains a maximum or minimum value along the constraint curve $g(x, y) = k$ at the point (a, b) and $\nabla g(a, b) \neq \mathbf{0}$, then there is a number λ satisfying

$$\nabla f(a, b) = \lambda \nabla g(a, b).$$

The number λ is called a **Lagrange multiplier**. A point (a, b) on the constraint curve satisfying this equation for some λ is called a **constrained critical point** of f .

The hottest and coldest points on the plate (revisited). On Friday we found the hottest and coldest points of the temperature function $T(x, y) = 2x^2 + y^2 - y$ on the plate $x^2 + y^2 \leq 1$. Step 1 gave the only interior critical point, $(0, \frac{1}{2})$, where $T = -\frac{1}{4}$. We will optimize on the boundary using Lagrange.

THE BOUNDARY CIRCLE $x^2 + y^2 = 1$

THE METHOD OF LAGRANGE MULTIPLIERS

To find the maximum and minimum values of $f(x, y)$ subject to the constraint $g(x, y) = k$ (assuming these extreme values exist and $\nabla g \neq \mathbf{0}$ on the constraint curve):

Step 1. Find every constrained critical point: all values of x , y , and λ such that

$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = k.$$

Step 2. Evaluate f at each point (x, y) found in Step 1. The largest of these values is the maximum value of f ; the smallest is the minimum.

This method works verbatim in higher dimensions, e.g. for $f(x, y, z)$ subject to $g(x, y, z) = k$.

Activity 1 _____ **A soda can**

A cylindrical soda can holds about 355 cc of liquid. In this activity, we want to find the dimensions of such a can that will minimize the surface area. For the sake of simplicity, assume the can is a perfect cylinder.

- (a) What are the variables in this problem? Based on the context, what restriction(s), if any, are there on these variables?

- (b) What quantity do we want to optimize in this problem? What equation describes the constraint? (You need to decide which of these functions plays the role of f and which plays the role of g in our discussion of Lagrange multipliers.)

- (c) Find λ and the values of your variables that satisfy $\nabla f = \lambda \nabla g$ in the context of this problem.

- (d) Determine the dimensions of the pop can that give the desired solution to this constrained optimization problem.

Activity 2 Lagrange multipliers at the board

At the board with your group. *I will assign the groups.* Pick a **scribe**, a **verifier**, and a **reporter**; the reporter cannot be the scribe.

GROUP WORK — AT THE BOARD

FIND YOUR GROUP BELOW.

Groups 1, 5 $f(x, y) = x^2 + y^2$ subject to $x^4 + y^4 = 16$.

Groups 2, 6 $f(x, y) = x + 2y$ subject to $x^2 + y^2 = 5$.

Groups 3, 7 $f(x, y, z, t) = x + y + z + t$ subject to $x^2 + y^2 + z^2 + t^2 = 4$.

Groups 4, 8 $f(x, y, z) = x^2 y^2 z^2$ subject to $x^2 + y^2 + z^2 = 3$.

- Write down the Lagrange system: $\nabla f = \lambda \nabla g$ and the constraint.
- Find *all* of its solutions, the constrained critical points. Be careful when you divide.
- Evaluate f at each one.

Box the maximum and minimum values of f and the points where they occur.

RECORD YOUR GROUP'S ANSWERS BELOW

WHICH ONE IS THE MAXIMUM?

The method produces constrained critical points; it does not say which is which. If the constraint set is closed and bounded (a circle, an ellipse, a sphere), the Extreme Value Theorem guarantees a maximum and a minimum, so the largest value among them is the maximum and the smallest is the minimum. If it is not, there may be no maximum or minimum at all, and you have to argue from the context. In Activity 1 the constraint curve $\pi r^2 h = 355$ is unbounded, so the Extreme Value Theorem says nothing. The surface area grows without bound as $r \rightarrow 0$ and as $r \rightarrow \infty$, so the single constrained critical point we found is the minimum, and there is no maximum.

Exit ticket

Name: _____

Complete this and turn it in at the end of class; it counts toward Engagement. **You may discuss with the people around you.**

Find the absolute maximum and absolute minimum values of

$$f(x, y) = x^2 - y^2$$

subject to the constraint $x^2 + y^2 = 9$, and the points where they occur.

Activity 1 is Activity 10.8.2 of *Active Calculus — Multivariable* (Schlicker, Keller, Long), activecalculus.org, CC BY-SA 4.0; Activity 2 adapts Exercises 10.8.3.3, 10.8.3.1, 10.8.3.9, and 10.8.3.8 (Groups 1, 5; 2, 6; 3, 7; 4, 8) from the same text, with smaller constants, and the exit ticket is Exercise 10.8.3.4. Lagrange's Theorem and the method box follow Stewart §14.8.