

Optimization

Name: _____

Stewart §14.7.

Preview debrief _____ **Local maxima of functions of two variables**

In Preview 7, f has a local maximum at (x_0, y_0) : the value $f(x_0, y_0)$ is greater than $f(x, y)$ for all (x, y) nearby.

PREVIEW 7**CRITICAL POINT**

A **critical point** of f is a point (x_0, y_0) in the domain of f at which $\nabla f(x_0, y_0) = \mathbf{0}$, or at which $\nabla f(x_0, y_0)$ does not exist (that is, f_x or f_y fails to exist there).

Key Fact: Local maxima and minima can only occur at critical points.

Activity 1 _____ **Finding critical points**

Complete with the people around you. Find the critical points of each of the following functions. We will then look at the graphs of the surfaces near each critical point and compare them to your work.

(a) $f(x, y) = 2 + x^2 - y^2$

(b) $f(x, y) = |x| + |y|$

(c) $f(x, y) = 2xy - 4x + 2y - 3$

THE HESSIAN

The **Hessian** of f is the matrix of second-order partial derivatives

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}, \quad \text{and its determinant is} \quad D = \det H = f_{xx}f_{yy} - f_{xy}^2,$$

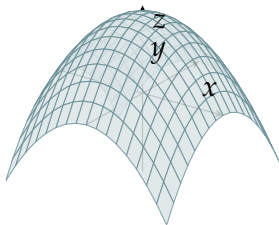
since $f_{xy} = f_{yx}$.

Three examples. Each of the functions

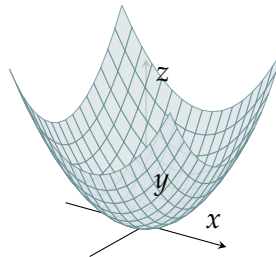
$$f_1(x, y) = 4 - x^2 - y^2, \quad f_2(x, y) = x^2 + y^2, \quad f_3(x, y) = x^2 - y^2$$

has a critical point at the origin. Their graphs are shown in Figure 1.

$$z = 4 - x^2 - y^2$$



$$z = x^2 + y^2$$



$$z = x^2 - y^2$$

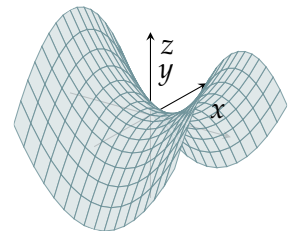


Figure 1. The graphs of f_1 , f_2 , and f_3 .

THE SECOND DERIVATIVE TEST

Suppose (a, b) is a critical point of f with $\nabla f(a, b) = \mathbf{0}$, and let

$$D = \det H(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2$$

be the determinant of the Hessian at (a, b) .

- If $D > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum at (a, b) .
- If $D > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum at (a, b) .
- If $D < 0$, then f has a saddle point at (a, b) .
- If $D = 0$, then the test gives no information about what happens at (a, b) .

Activity 2 Finding and classifying critical points

At the board with your group. *I will assign the groups.* Roles by **last** name in alphabetical order: first is **scribe**, second is **verifier**, third is **reporter**.

GROUP WORK — AT THE BOARD

FIND YOUR GROUP BELOW.

Groups 1, 5 $f(x, y) = 3x^3 + y^2 - 9x + 4y$

Groups 2, 6 $f(x, y) = xy + \frac{2}{x} + \frac{4}{y}$

Groups 3, 7 $f(x, y) = x^3 + y^3 - 3xy$

Groups 4, 8 $f(x, y) = \frac{1}{x} + \frac{2}{y} + 3x + 4y$, for $x, y > 0$

- Find all critical points of your function.
- Use the Second Derivative Test to classify each critical point as a local maximum, a local minimum, or a saddle point.

Box your critical points and their classifications.

Fill in the rest of the table during the discussion.

Groups	Function	Critical points	Classification
1, 5	$3x^3 + y^2 - 9x + 4y$		
2, 6	$xy + \frac{2}{x} + \frac{4}{y}$		
3, 7	$x^3 + y^3 - 3xy$		
4, 8	$\frac{1}{x} + \frac{2}{y} + 3x + 4y$		

Optimization on a restricted domain _____ The Extreme Value Theorem

RECALL: ONE VARIABLE

ABSOLUTE EXTREMA, CLOSED AND BOUNDED REGIONS

An **absolute maximum (minimum)** of f on a region R is a largest (smallest) value of f over the whole region. A **closed** region is one that contains its boundary. The meaning of a **bounded** region should be intuitively clear.

CLOSED? BOUNDED?

THE EXTREME VALUE THEOREM

Let $f = f(x, y)$ be a continuous function on a closed and bounded region R . Then f has an absolute maximum and an absolute minimum in R . They must occur either at a critical point in the interior of R or at a boundary point of R .

Example. Suppose the temperature T at each point on the circular plate $x^2 + y^2 \leq 1$ is given by $T(x, y) = 2x^2 + y^2 - y$. We will find the hottest and coldest points on the plate.

THE HOTTEST AND COLDEST POINTS ON THE PLATE**ABSOLUTE EXTREMA ON A CLOSED AND BOUNDED REGION**

- Step 1.** Find the critical points of f in the interior of the region.
- Step 2.** Parametrize the boundary, for example as $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, $a \leq t \leq b$.
- Step 3.** Find the critical points of f on the boundary, using methods from Calculus I on the single-variable function $f(x(t), y(t))$. Include the endpoints of the interval.
- Step 4.** Evaluate f at all of the critical points from the preceding steps.
- Step 5.** Identify the largest and smallest values.

Exit ticket

Name: _____

Complete this and turn it in at the end of class; it counts toward Engagement. **You may discuss with the people around you.**

Find the absolute maximum and absolute minimum values of

$$f(x, y) = x^3 - 3x + y^2$$

on the rectangle $R = \{(x, y) : 0 \leq x \leq 2, -1 \leq y \leq 1\}$, and the points where they occur.

Activities 1–2 are Activities 10.7.2 and 10.7.4 of *Active Calculus — Multivariable* (Schlicker, Keller, Long), activecalculus.org, CC BY-SA 4.0; Activity 1 is adapted, and Activity 2 adds the function of Exercise 10.7.5.4 to Activity 10.7.4. The three examples before the Second Derivative Test are from Activity 10.7.3, and the Second Derivative Test, the procedure for closed and bounded regions, and the example are from the same section (Example 10.7.8).