

Directional Derivatives and the Gradient

Name: _____

Stewart §14.6.

Preview debrief _____ **Walking through a temperature field**

In Preview 6, $f(x, y) = 30 - x^2 - \frac{1}{2}y^2$ is the temperature, in degrees Celsius, at the point (x, y) , where x and y are measured in feet. We discuss parts (d) and (e).

PREVIEW 6, PARTS (D) AND (E)**DIRECTIONAL DERIVATIVES**

DIRECTIONAL DERIVATIVE

The **directional derivative** of f at (a, b) in the direction of a *unit* vector $\mathbf{u} = \langle u_1, u_2 \rangle$ is

$$D_{\mathbf{u}}f(a, b) = \lim_{h \rightarrow 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h},$$

provided the limit exists. It measures the instantaneous rate of change of f at (a, b) in the direction of $\mathbf{u}$.

COMPUTING $D_{\mathbf{u}}f$ WITH THE CHAIN RULE**THE GRADIENT**

If f is a function of two variables, the **gradient** of f is the vector function

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle, \quad \text{or more compactly} \quad \nabla f = \langle f_x, f_y \rangle.$$

If f is differentiable, then for every unit vector $\mathbf{u}$,

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}, \quad \text{or} \quad D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}.$$

Activity 1 _____ **Computing directional derivatives**

Let $f(x, y) = 3xy - x^2y^3$.

- (a) Determine $f_x(x, y)$ and $f_y(x, y)$.

- (b) Use the formula $D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$ to determine $D_{\mathbf{i}}f(x, y)$ and $D_{\mathbf{j}}f(x, y)$. What familiar function is $D_{\mathbf{i}}f$? What familiar function is $D_{\mathbf{j}}f$? (Recall that $\mathbf{i}$ is the unit vector in the positive x -direction and $\mathbf{j}$ is the unit vector in the positive y -direction.)
- (c) Use this formula to find the derivative of f in the direction of the vector $\mathbf{v} = \langle 2, 3 \rangle$ at the point $(1, -1)$. Remember that a unit direction vector is needed.

Activity 2

The gradient on a contour plot

Let's consider the function f defined by $f(x, y) = x^2 - y^2$. Some contours for this function are shown in Figure 1.

- (a) Find the gradient $\nabla f(x, y)$.

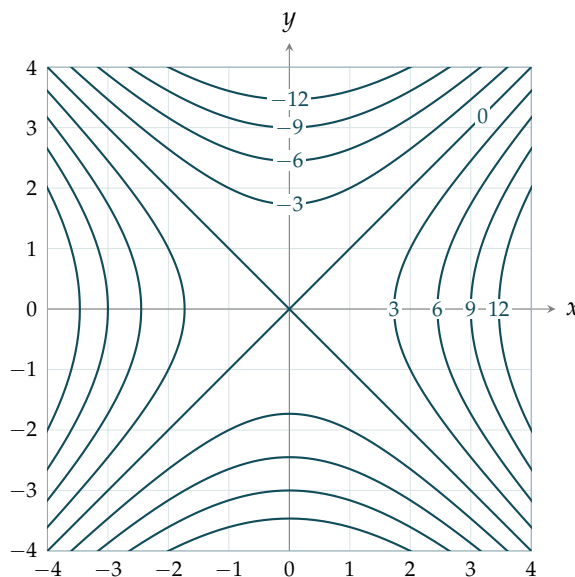


Figure 1. A contour plot of $f(x, y) = x^2 - y^2$.

- (b) For each of the following points (x_0, y_0) , evaluate the gradient $\nabla f(x_0, y_0)$ and sketch the gradient vector with its tail at (x_0, y_0) . Some of the vectors are too long to fit onto the plot, but we'd like to draw them to scale; to do so, scale each vector by a factor of $\frac{1}{4}$.

(x_0, y_0)	$(2, 0)$	$(0, 2)$	$(2, 2)$	$(2, 1)$	$(-3, 2)$	$(-2, -4)$	$(0, 0)$
$\nabla f(x_0, y_0)$							

- (c) What do you notice about the relationship between the gradient at (x_0, y_0) and the contour passing through that point?
- (d) Does f increase or decrease in the direction of $\nabla f(x_0, y_0)$? Provide a justification for your response.

THE GRADIENT IS PERPENDICULAR TO LEVEL CURVES

THE GRADIENT AND LEVEL CURVES

If f is differentiable and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is perpendicular to the level curve $f(x, y) = k$ through (x_0, y_0) , where $k = f(x_0, y_0)$.

LEVEL SURFACES

The same idea works for functions of three variables. If $f(x, y, z)$ is differentiable, its gradient is $\nabla f = \langle f_x, f_y, f_z \rangle$, and $\nabla f(x_0, y_0, z_0)$ is perpendicular to the level surface $f(x, y, z) = k$ through (x_0, y_0, z_0) , where $k = f(x_0, y_0, z_0)$.

TANGENT PLANES REVISITED

Activity 3 _____ **The direction of greatest increase**

In this activity we investigate how the gradient is related to the directions of greatest increase and decrease of a function. Let f be a differentiable function and $\mathbf{u}$ a unit vector.

- (a) Let θ be the angle between $\nabla f(x_0, y_0)$ and $\mathbf{u}$. Write the dot product $\nabla f(x_0, y_0) \cdot \mathbf{u}$ in terms of the lengths of the two vectors and θ . Then simplify, using the fact that $\mathbf{u}$ is a unit vector.

$$D_{\mathbf{u}}f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u} = \underline{\hspace{2cm}}$$

- (b) As we change the direction $\mathbf{u}$, the length $|\nabla f(x_0, y_0)|$ stays the same, and only θ changes. Which value of θ makes $D_{\mathbf{u}}f(x_0, y_0)$ as large as possible? Why?

- (c) For this value of θ , which way does $\mathbf{u}$ point, relative to $\nabla f(x_0, y_0)$? This is the direction of greatest increase.

- (d) Which value of θ makes $D_{\mathbf{u}}f(x_0, y_0)$ as small as possible? Which way does $\mathbf{u}$ point then, relative to $\nabla f(x_0, y_0)$? This is the direction of greatest decrease.
- (e) Write the unit vectors $\mathbf{u}$ and $\mathbf{v}$, in terms of $\nabla f(x_0, y_0)$, that point in the directions of greatest increase and greatest decrease of f at (x_0, y_0) . What must be true of $\nabla f(x_0, y_0)$ for these vectors to exist?

THE MAGNITUDE OF THE GRADIENT

SUMMARY: THE DIRECTION AND RATE OF GREATEST INCREASE

Suppose f is differentiable and $\nabla f(x_0, y_0) \neq \mathbf{0}$. The direction of greatest increase of f at (x_0, y_0) is $\nabla f(x_0, y_0)$. A unit vector in that direction is $\mathbf{u} = \frac{1}{|\nabla f(x_0, y_0)|} \nabla f(x_0, y_0)$. The maximum rate of change is $D_{\mathbf{u}}f(x_0, y_0) = |\nabla f(x_0, y_0)|$.

Likewise, the direction of greatest decrease is $-\nabla f(x_0, y_0)$. A unit vector in that direction is $-\mathbf{u}$. The minimum rate of change is $-|\nabla f(x_0, y_0)|$.

Exit ticket

Name: _____

Complete this and turn it in at the end of class; it counts toward Engagement. **You may discuss with the people around you.**

Let $f(x, y) = x^2 + 3y^2$.

- (a) Find $\nabla f(x, y)$ and $\nabla f(1, 2)$.

- (b) Find the direction of greatest increase in f at the point $(1, 2)$. Explain.

- (c) Find a direction $\mathbf{w}$ for which the derivative of f in the direction of $\mathbf{w}$ is zero.

Activities 1–3 are Activities 10.6.2, 10.6.3, and 10.6.4 of *Active Calculus — Multivariable* (Schlicker, Keller, Long), active-calculus.org, CC BY-SA 4.0. The exit ticket is parts (a), (b), and (d) of Exercise 10.6.8.14 from the same text, without the figures. The figure in Activity 2 is redrawn, and Activity 3 is adapted.