

The Chain Rule and Directional Derivatives

Name: _____

Stewart §14.5 and §14.6 (directional derivatives).

Warm-up _____ **Moving along the tangent plane**

Let $f(x, y) = x^2y^3$. Recall that the linearization of f at (a, b) is given by $L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$. Consider the parametric curve $x(t) = e^t$, $y(t) = \cos(t) + t$, which passes through $(1, 1)$ when $t = 0$.

(a) Find the linearization $L(x, y)$ of f at $(1, 1)$.

(b) Find $\frac{d}{dt} [L(x(t), y(t))] \text{ at } t = 0$.

(c) Substitute to write $z = f(x(t), y(t))$ as a function of t , and find $\frac{dz}{dt}$ at $t = 0$.

THE CHAIN RULE (CASE 1)

Suppose that $z = f(x, y)$ is a differentiable function of x and y , where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t . Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

Activity 1 **Trying the Chain Rule**

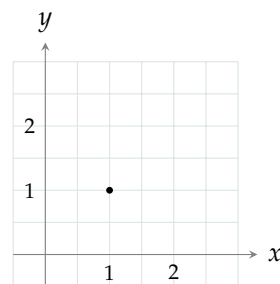
With the person next to you. Use the Chain Rule to find $\frac{dz}{dt}$ at the given value of t .

(a) $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$; $t = 0$.

(b) $z = e^{xy}$, where $x = t^2$ and $y = \frac{1}{t}$; $t = 1$. Then check your answer by substituting. Which way was easier?

(c) $z = \ln(x^2 + y^2)$, where $x = e^t$ and $y = e^{-t}$; $t = 0$.

- (d) Find the level curve of f that passes through $(1, 1)$, and sketch it. On the same axes, sketch the path $(x(t), y(t))$ for t near 0. (Hint: what is xy ?) Then draw the vectors $\langle x'(0), y'(0) \rangle$ and $\langle f_x(1, 1), f_y(1, 1) \rangle$ starting at $(1, 1)$. Use your picture to explain your answer to (c).



THE CHAIN RULE (GENERAL VERSION)

Suppose that u is a differentiable function of the n variables $x_1, x_2, \dots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \dots, t_m$. Then u is a function of $t_1, \dots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \dots, m$.

EXAMPLE: $z = e^x \sin y$, WHERE $x = st^2$ AND $y = s^2t$

TREE DIAGRAMS

To find $\frac{\partial u}{\partial t_i}$ in the General Version:

1. Draw the tree: put u at the top, draw a branch from u down to each of $x_1, \dots, x_n$, and draw a branch from each x_j down to each of $t_1, \dots, t_m$.
2. Label the branch from u to x_j with $\frac{\partial u}{\partial x_j}$, and the branch from x_j to t_i with $\frac{\partial x_j}{\partial t_i}$.
3. For each path from u down to t_i , multiply the labels along the path.
4. Add the products from all of these paths.

EXAMPLE: $w = f(x, y, z)$, WHERE x, y , AND z ARE FUNCTIONS OF r AND s

IMPLICIT DIFFERENTIATION

Activity 2 _____ Implicit differentiation

Use the formula above to find $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy$.

DIRECTIONAL DERIVATIVES

DIRECTIONAL DERIVATIVE

The **directional derivative** of f at (a, b) in the direction of a *unit* vector $\mathbf{u} = \langle u_1, u_2 \rangle$ is

$$D_{\mathbf{u}}f(a, b) = \lim_{h \rightarrow 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h},$$

provided the limit exists. It measures the instantaneous rate of change of f at (a, b) in the direction of $\mathbf{u}$.

COMPUTING $D_{\mathbf{u}}f$ WITH THE CHAIN RULE**THE GRADIENT**

If f is a function of two variables, the **gradient** of f is the vector function

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle.$$

If f is differentiable, then for every unit vector $\mathbf{u}$,

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

EXAMPLE: $f(x, y) = x^2y^3 - 4y$ AT $(2, -1)$ IN THE DIRECTION OF $\mathbf{v} = \langle 2, 5 \rangle$