

Second Partial Derivatives and Tangent Planes

Name: _____

Stewart §14.3 (second partial derivatives) and §14.4.

SECOND PARTIAL DERIVATIVES

If f is a function of two variables, then there are four second-order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = (f_x)_x = f_{xx}, \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = (f_x)_y = f_{xy},$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = (f_y)_x = f_{yx}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = (f_y)_y = f_{yy}.$$

EXAMPLE: $f(x, y) = x^3 y^2 - 4xy + e^{2y}$

Activity 1 _____ Which order is easier?

With the person next to you. Let

$$f(x, y) = x^2 \cos y + \frac{e^y}{1 + y^2}.$$

(a) Find f_{xy} by computing f_x first.(b) Find f_{yx} by computing f_y first.

(c) Which was easier, and why? What do you notice about your two answers?

Done early? Let $g(x, y) = xy^3 + \frac{\ln x}{x^2 + 1}$. Find the mixed partial both ways. Which is easier now?

CLAIRAUT'S THEOREM

Suppose f is defined on a disk D that contains the point (a, b) . If the functions f_{xy} and f_{yx} are both continuous on D , then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

The same holds for higher derivatives: for example, $f_{xyy} = f_{yyx} = f_{yxy}$ when these are continuous.

TANGENT PLANE

Let P be a point on the surface $z = f(x, y)$. The **tangent plane** to the surface at P , if it exists, is the plane that contains the tangent lines at P to all curves on the surface through P .

NORMAL VECTOR FOR THE TANGENT PLANE

EQUATION OF THE TANGENT PLANE

Suppose f has continuous partial derivatives. The **tangent plane** to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Activity 2 **Back to the wind chill**

With the person next to you. Recall the wind chill $w(v, T)$ from Worksheet 12. We know $w(15, -10) = -32$, and from the Worksheet 12 warm-up, $w_v(15, -10) \approx -0.6$ °F per mi/h.

$v \backslash T$	-20	-15	-10	-5	0
5	-34	-28	-22	-16	-11
10	-41	-35	-28	-22	-16
15	-45	-39	-32	-26	-19
20	-48	-42	-35	-29	-22
25	-51	-44	-37	-31	-24
30	-53	-46	-39	-33	-26

(a) Estimate $w_T(15, -10)$. What does it mean?

(b) Estimate $w(13, -10)$. (Recall that for a function f of one variable, $f(x) \approx L(x) = f(a) + f'(a)(x - a)$ for x near a .)

(c) Estimate $w(15, -12)$.

(d) How would you combine your results to estimate $w(13, -12)$?

LINEAR APPROXIMATION

The **linearization** of f at (a, b) (or the **linear approximation** of f) is the function

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b),$$

whose graph is the tangent plane to the surface $z = f(x, y)$ at $(a, b, f(a, b))$.

Activity 3 _____ **Tangent planes and approximations**

For (a) and (b), find the tangent plane to $z = f(x, y)$ at the given point, and use the linearization $L(x, y)$ to estimate the given value.

(a) $f(x, y) = \sqrt{x^2 + y^2}$ at $(3, 4)$. Estimate $f(2.9, 4.1)$ (don't use a calculator).

(b) $f(x, y) = e^x \cos y$ at $(0, 0)$. Estimate $f(0.1, 0.1)$ (don't use a calculator).

(c) Let $f(x, y) = 4 - x - y$. Without computing anything, what is the tangent plane to $z = f(x, y)$ at any point? How good is the linear approximation of f ?

Done early? Consider the surface $z = x^2 + y^2$.

- (i) Find parametric equations for the **normal line** to the surface at an arbitrary point $(a, b, a^2 + b^2)$: the line through that point perpendicular to the tangent plane.

- (ii) The closest point on the surface to $Q = (0, 0, \frac{5}{2})$ has a normal line that passes through Q . Use this to find the closest point(s).

DIFFERENTIABILITY

We call f **differentiable** at (a, b) when the tangent plane at (a, b) is a good approximation of f near (a, b) (Stewart gives the precise definition).

PARTIAL DERIVATIVES ARE NOT ENOUGH: $g(x, y) = \frac{x^2 y}{x^2 + y^2}$, $g(0, 0) = 0$

However, if the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b) , then one can prove that the tangent plane is a good approximation of f near (a, b) , so f is differentiable at (a, b) .

Exit ticket

Name: _____

Complete this and turn it in at the end of class; it counts toward Engagement. **You may discuss with the people around you.**

(a) Let $f(x, y) = x \sin y + e^{y^2}$. Find f_{xy} . Choose your order wisely.

(b) Let $f(x, y) = x^2 y^3$. Find the tangent plane to the surface $z = f(x, y)$ at $(1, 1, 1)$, and use it to estimate $f(1.02, 0.97)$ (don't use a calculator).