

Limits and Continuity

Name: _____

Stewart §14.2.

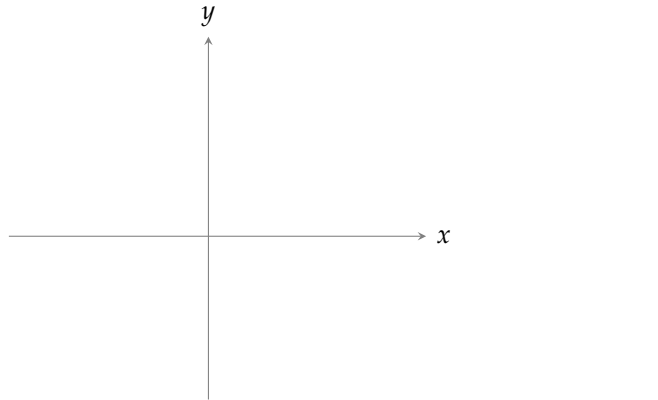
THE LIMIT OF A FUNCTION OF TWO VARIABLES

We write

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$$

if we can make the values of $f(x,y)$ as close to L as we like by taking (x,y) in the domain of f sufficiently close (but not equal) to (a,b) .

DEFINITION OF LIMIT



THE TWO-PATH TEST

EXAMPLE: THE PREVIEW, $g(x,y) = \frac{2xy}{x^2 + y^2}$

Activity 1 Testing paths

At the board with your group. *I will assign the groups.* Roles by **first** name in **alphabetical** order: first is **scribe**, second is **verifier**, third is **reporter**.

GROUP WORK — AT THE BOARD

Find your group number below. Each limit is as $(x, y) \rightarrow (0, 0)$.

Groups 1, 5 $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$

Groups 2, 6 $f(x, y) = \frac{x^2 + xy}{x^2 + y^2}$

Groups 3, 7 $f(x, y) = \frac{xy^2}{x^2 + y^4}$

Groups 4, 8 $f(x, y) = \frac{x^2y}{x^2 + y^2}$

On the board, large enough to read from across the room:

1. The x -axis is parametrized by $\mathbf{r}_1(t) = \langle t, 0 \rangle$ and the y -axis by $\mathbf{r}_2(t) = \langle 0, t \rangle$. Compute $\lim_{t \rightarrow 0} f(\mathbf{r}_1(t))$ and $\lim_{t \rightarrow 0} f(\mathbf{r}_2(t))$.
2. Find a parametrization $\mathbf{r}(t)$ of the line $y = mx$ with $\mathbf{r}(0) = \langle 0, 0 \rangle$, and compute $\lim_{t \rightarrow 0} f(\mathbf{r}(t))$. Your answer may depend on m .
3. Do the same for the parabola $y = x^2$ and the parabola $x = y^2$.
4. Verdict: does $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ exist, not exist, or can you not tell yet? Explain.

Done early? Groups 1, 2, 5, 6: find a path through the origin along which f approaches $\frac{1}{2}$. Groups 3, 7: find one along which f approaches $\frac{2}{5}$. Groups 4, 8: can you find any path along which f approaches something other than 0?

Record every group's results. Under each path, write the limit along that path (for $y = mx$, your answer in terms of m). Under *Verdict*, write the limit if it exists, *DNE* if it does not, or *can't tell*.

Limit as $(x, y) \rightarrow (0, 0)$ along

Groups	x -axis	y -axis	$y = mx$	$y = x^2$	$x = y^2$	Verdict
1, 5						
2, 6						
3, 7						
4, 8						

THE SQUEEZE THEOREM

If $|f(x, y) - L| \leq h(x, y)$ for all (x, y) near (a, b) , except possibly at (a, b) , and $\lim_{(x, y) \rightarrow (a, b)} h(x, y) = 0$, then $\lim_{(x, y) \rightarrow (a, b)} f(x, y) = L$.

USING THE SQUEEZE THEOREM: $\lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 y}{x^2 + y^2}$

LIMIT LAWS

Let $f = f(x, y)$ and $g = g(x, y)$ be functions of two variables. In each line, $\lim$ means $\lim_{(x, y) \rightarrow (a, b)}$, and $\lim f$ and $\lim g$ are assumed to exist.

1. $\lim cf = c \lim f$ for any constant c
2. $\lim(f \pm g) = \lim f \pm \lim g$
3. $\lim fg = (\lim f)(\lim g)$
4. $\lim \frac{f}{g} = \frac{\lim f}{\lim g}$, provided $\lim g \neq 0$

CONTINUITY

A function f of two variables is **continuous at** (a, b) if

$$\lim_{(x, y) \rightarrow (a, b)} f(x, y) = f(a, b).$$

We say f is continuous on D if f is continuous at every point of D .

KEY FACTS ABOUT CONTINUITY

Activity 2 _____ Change of variables

With the person next to you.

(a) Let $S(u) = \frac{\sin u}{u}$ for $u \neq 0$, and $S(0) = 1$.

(i) Explain why S is continuous at $u = 0$.

(ii) Explain why $h(x, y) = S(x^2 + y^2)$ is continuous on all of $\mathbb{R}^2$.

(iii) For $(x, y) \neq (0, 0)$, compare $\frac{\sin(x^2 + y^2)}{x^2 + y^2}$ with $h(x, y)$. Use this to find $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2}$.

Activity 3 _____ Does the limit exist?

With the person next to you. For each limit below, decide whether it exists. If it does, find it and justify your answer; if it does not, show why not.

Today we have seen four methods: **testing paths**, **simplifying algebraically**, **squeezing**, and **changing variables**. Each problem below can be solved with one of them, and each method is used exactly once.

(a) $\lim_{(x,y) \rightarrow (0,0)} \frac{e^{x^2+y^2} - 1}{x^2 + y^2}$

(b) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^4 - y^4}{x^2 + y^2}$

(c) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + y^3}{x^2 + y^2}$

(d) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + y^2}$

Done early? For which integers $n \geq 0$ does $\lim_{(x,y) \rightarrow (0,0)} \frac{x^n y}{x^2 + y^2}$ exist?

Exit ticket

Name: _____

Complete this and turn it in at the end of class; it counts toward Engagement. **You may discuss with the people around you.**

For each limit, decide whether it exists. If it does, find it and justify your answer; if it does not, show why not.

(a) $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^3}{x^2 + y^6}$

(b) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 \sin y}{x^2 + y^2}$