

Name: Answer Key

15 minutes

This quiz is out of 12 points: Problem 1 is worth 8 points (4 points for each part) and Problem 2 is worth 4 points. No calculators, notes, or other materials allowed. Show your work and explain your reasoning, I want to give you partial credit!

Problem 1. One of these limits exists and one does not. Determine which is which, and show it.

(a) $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + 2y^2}$

Solution. Let $f(x, y) = \frac{xy}{x^2 + 2y^2}$. To approach $(0, 0)$ along a path, we parametrize the path by a curve $\mathbf{r}(t)$ with $\mathbf{r}(0) = (0, 0)$, substitute it into f , and take the resulting Calc I limit as $t \rightarrow 0$. We use two paths.

Path 1: the x -axis, $\mathbf{r}_1(t) = \langle t, 0 \rangle$. For $t \neq 0$,

$$f(\mathbf{r}_1(t)) = f(t, 0) = \frac{t \cdot 0}{t^2 + 2 \cdot 0^2} = \frac{0}{t^2} = 0, \quad \text{so} \quad \lim_{t \rightarrow 0} f(\mathbf{r}_1(t)) = \lim_{t \rightarrow 0} 0 = 0.$$

Path 2: the line $y = x$, $\mathbf{r}_2(t) = \langle t, t \rangle$. For $t \neq 0$,

$$f(\mathbf{r}_2(t)) = f(t, t) = \frac{t \cdot t}{t^2 + 2t^2} = \frac{t^2}{3t^2} = \frac{1}{3}, \quad \text{so} \quad \lim_{t \rightarrow 0} f(\mathbf{r}_2(t)) = \lim_{t \rightarrow 0} \frac{1}{3} = \frac{1}{3}.$$

Since we found two paths along which f has different limits, **this limit does not exist.**

(b) $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$

Solution. Let $g(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}$.

Guess the limit. Along the x -axis, $\mathbf{r}_1(t) = \langle t, 0 \rangle$. For $t \neq 0$,

$$g(\mathbf{r}_1(t)) = g(t, 0) = \frac{t \cdot 0}{\sqrt{t^2 + 0^2}} = 0, \quad \text{so} \quad \lim_{t \rightarrow 0} g(\mathbf{r}_1(t)) = \lim_{t \rightarrow 0} 0 = 0.$$

So if the limit exists, it must be 0. We now prove that it is 0 using the Squeeze Theorem.

Prove it. For $(x, y) \neq (0, 0)$ we have $x^2 \leq x^2 + y^2$, so $\frac{x^2}{x^2 + y^2} \leq 1$, and therefore its square root is at most $\sqrt{1} = 1$. Writing $|x| = \sqrt{x^2}$ and combining the square roots,

$$\begin{aligned} 0 \leq \left| \frac{xy}{\sqrt{x^2 + y^2}} \right| &= |y| \cdot \frac{|x|}{\sqrt{x^2 + y^2}} = |y| \cdot \frac{\sqrt{x^2}}{\sqrt{x^2 + y^2}} \\ &= |y| \cdot \sqrt{\frac{x^2}{x^2 + y^2}} \leq |y| \cdot 1 = |y|. \end{aligned}$$

As $(x, y) \rightarrow (0, 0)$, $|y| \rightarrow 0$, so by the Squeeze Theorem

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2 + y^2}} = 0.$$

This limit exists.

Problem 2. Let $f(x, y) = x^3y^2 + \frac{e^y \sin y}{1 + y^4}$. Find f_{xx} , f_{yx} , and f_{yyx} .

Solution. The second summand is a function of y only, so

$$f_x = 3x^2y^2 \quad \text{and} \quad \boxed{f_{xx} = 6xy^2}.$$

The function f has continuous second order partial derivatives because it is built from polynomials, e^y , and $\sin y$ using sums, products, and a quotient whose denominator $1 + y^4$ is never 0. By Clairaut's Theorem, $f_{yx} = f_{xy}$. Thus,

$$f_{yx} = f_{xy} = (f_x)_y = \frac{\partial}{\partial y}(3x^2y^2) = \boxed{6x^2y}.$$

(Computing f_y first requires the quotient rule on $\frac{e^y \sin y}{1+y^4}$, but that term does not involve x , so it disappears when we then differentiate with respect to x .)

The same reasoning applies to third-order partials, since they are continuous everywhere: $f_{yyx} = f_{xyy}$. So

$$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y}(6x^2y) = 6x^2, \quad \text{so} \quad \boxed{f_{yyx} = 6x^2}.$$