

Name: Answer Key

15 minutes

This quiz is out of 12 points: Problems 1, 2, and 3 are each worth 4 points. No calculators, notes, or other materials allowed. Show your work and explain your reasoning, I want to give you partial credit!

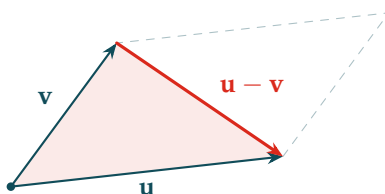
Problem 1. Let $\mathbf{u} = \langle 1, 2, -1 \rangle$ and $\mathbf{v} = \langle 3, 0, 2 \rangle$. Compute $\mathbf{u} \times \mathbf{v}$, and then find the area of the triangle having $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} - \mathbf{v}$ as its sides.

Solution. Compute the cross product using the determinant formula:

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 3 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \mathbf{k} \\ &= 4\mathbf{i} - 5\mathbf{j} + (-6)\mathbf{k} = \boxed{\langle 4, -5, -6 \rangle}.\end{aligned}$$

The area of the triangle is half the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$. Thus, the area is given by

$$A = \frac{1}{2} |\mathbf{u} \times \mathbf{v}| = \frac{1}{2} \sqrt{4^2 + (-5)^2 + (-6)^2} = \frac{1}{2} \sqrt{16 + 25 + 36} = \boxed{\frac{\sqrt{77}}{2}}.$$



Problem 2. Let $P = (2, 1, 0)$ and $Q = (4, 0, 3)$. Find parametric equations for the line through P and Q , and determine whether the point $(8, -2, 9)$ lies on this line.

Solution. A point on the line is $P = (2, 1, 0)$. One possibility for the direction of the line is:

$$\mathbf{v} = \overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} = \langle 2, 1, 0 \rangle - \langle 4, 0, 3 \rangle = \langle -2, 1, -3 \rangle.$$

A vector equation of the line using the above data is $\mathbf{r}(t) = \langle 2, 1, 0 \rangle + t\langle -2, 1, -3 \rangle$. The corresponding parametric equations are

$$\boxed{x = 2 - 2t, \quad y = 1 + t, \quad z = -3t.}$$

The point $(8, -2, 9)$ lies on the line if there exists a value of t satisfying $x(t) = 8$, $y(t) = -2$, and $z(t) = 9$. The z -equation $9 = -3t$ gives $t = -3$, and then

$$\mathbf{r}(-3) = \langle 2 + 6, 1 - 3, 9 \rangle = \langle 8, -2, 9 \rangle.$$

Yes, the point $(8, -2, 9)$ lies on the line.

Problem 3. Find an equation of the plane that contains the line $x = 1 + t$, $y = 2t$, $z = 3 - t$ and the point $(4, 0, 5)$.

Solution. Setting $t = 0$ shows that $(1, 0, 3)$ is a point on the line. From the parametric equations, the line has direction $\mathbf{u} = \langle 1, 2, -1 \rangle$. The plane is parallel to both $\mathbf{u}$ and the displacement vector from $(1, 0, 3)$ to $(4, 0, 5)$,

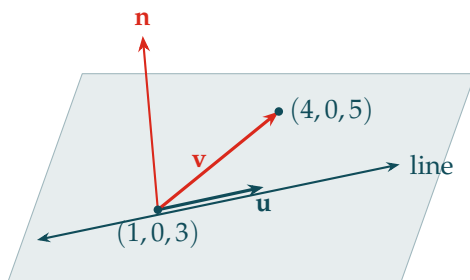
$$\mathbf{v} = \langle 4, 0, 5 \rangle - \langle 1, 0, 3 \rangle = \langle 3, 0, 2 \rangle.$$

These are the vectors from Problem 1, so

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \langle 4, -5, -6 \rangle$$

is a normal vector for the plane. Using the point $(4, 0, 5)$, an equation of the plane is

$$4(x - 4) + (-5)(y - 0) + (-6)(z - 5) = 0, \quad \text{or} \quad 4x - 5y - 6z = -14.$$



Challenge Problem. The lines

$$L_1 : x = t, y = 1 + t, z = 2 - t \quad \text{and} \quad L_2 : x = 1 + s, y = 2s, z = 3 + s$$

are skew. Find an equation of the plane that is parallel to both lines and lies halfway between them.

Solution. The line L_1 passes through $(0, 1, 2)$ with direction $\mathbf{d}_1 = \langle 1, 1, -1 \rangle$, and L_2 passes through $(1, 0, 3)$ with direction $\mathbf{d}_2 = \langle 1, 2, 1 \rangle$. A plane parallel to both lines has a normal vector which is orthogonal to both directions, so we take:

$$\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} = \langle 3, -2, 1 \rangle.$$

A plane with this normal vector has an equation of the form $3x - 2y + z = d$ for some d .

- The plane containing L_1 passes through $(0, 1, 2)$, so $d = 0 - 2 + 2 = 0$.
- The plane containing L_2 passes through $(1, 0, 3)$, so $d = 3 - 0 + 3 = 6$.

These are parallel planes with the same normal vector $\mathbf{n}$, so the plane halfway between them is found by taking the average $d = \frac{0+6}{2} = 3$:

$$3x - 2y + z = 3.$$