

Name: Answer Key

15 minutes

This quiz is out of 12 points: Problems 1, 2, and 3 are each worth 4 points. No calculators, notes, or other materials allowed. Show your work and explain your reasoning, I want to give you partial credit!

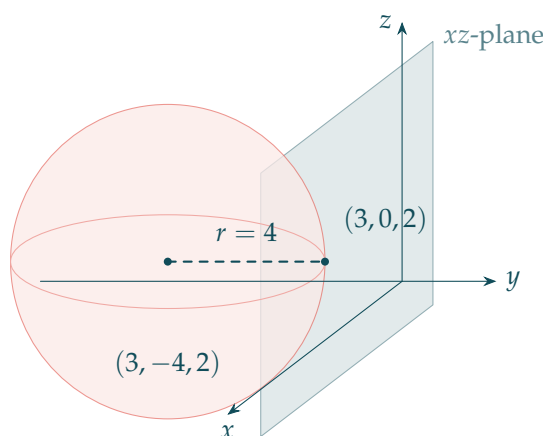
Problem 1. Find an equation of the sphere centered at $(3, -4, 2)$ that touches the xz -plane at exactly one point. *It may be helpful to draw a picture.*

Solution. We are given the center of the sphere and need to compute the radius r . The xz -plane is given by the equation $y = 0$. The projection of $A = (3, -4, 2)$ onto the xz -plane is the point $B = (3, 0, 2)$. In order for the sphere to touch the plane at exactly one point, the radius should be the distance between these two points:

$$r = |AB| = 4.$$

The sphere of radius 4 centered at $(3, -4, 2)$ is

$$(x - 3)^2 + (y + 4)^2 + (z - 2)^2 = 16.$$



Problem 2. Let $P = (1, 0, 2)$ and $Q = (3, -1, 4)$. Find a unit vector that points in the same direction as $\overrightarrow{QP}$.

Solution. Write $\overrightarrow{QP}$ as the difference of position vectors:

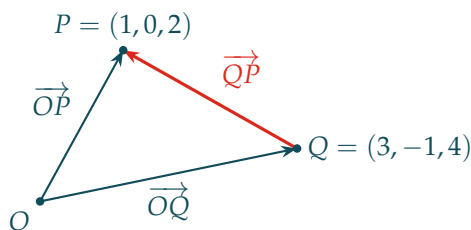
$$\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} = \langle 1, 0, 2 \rangle - \langle 3, -1, 4 \rangle = \langle -2, 1, -2 \rangle.$$

Its magnitude is

$$|\overrightarrow{QP}| = \sqrt{(-2)^2 + 1^2 + (-2)^2} = \sqrt{4 + 1 + 4} = 3.$$

Dividing by the magnitude gives a unit vector in the same direction:

$$\frac{\overrightarrow{QP}}{|\overrightarrow{QP}|} = \frac{1}{3} \langle -2, 1, -2 \rangle = \left\langle -\frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right\rangle.$$



Problem 3. Suppose $|\mathbf{u}| = 2$, $|\mathbf{v}| = 3$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\pi/3$. Find $|\mathbf{u} - 2\mathbf{v}|$.

Solution. Compute $|\mathbf{u} - 2\mathbf{v}|^2$ using dot product properties:

$$\begin{aligned} |\mathbf{u} - 2\mathbf{v}|^2 &= (\mathbf{u} - 2\mathbf{v}) \cdot (\mathbf{u} - 2\mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} - 2\mathbf{v} \cdot \mathbf{u} - 2\mathbf{u} \cdot \mathbf{v} + 4\mathbf{v} \cdot \mathbf{v} \\ &= |\mathbf{u}|^2 - 4\mathbf{u} \cdot \mathbf{v} + 4|\mathbf{v}|^2. \end{aligned}$$

Use the dot product formula:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta = 2 \cdot 3 \cdot \cos(\pi/3) = 6 \cdot \frac{1}{2} = 3.$$

Thus,

$$|\mathbf{u} - 2\mathbf{v}|^2 = 2^2 - 4(3) + 4 \cdot 3^2 = 4 - 12 + 36 = 28.$$

Finally,

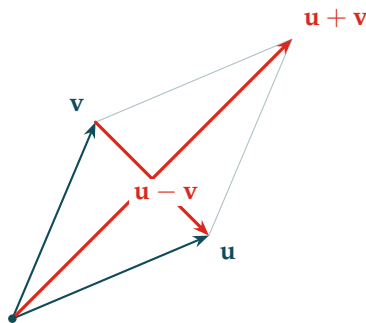
$$|\mathbf{u} - 2\mathbf{v}| = \sqrt{28} = 2\sqrt{7}.$$

Challenge Problem. Suppose $|\mathbf{u}| = |\mathbf{v}|$. Prove that $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal.

Solution. Two vectors are orthogonal if and only if their dot product is zero. Compute the dot product using dot product properties:

$$\begin{aligned} (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) &= \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{v} \\ &= |\mathbf{u}|^2 + \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} - |\mathbf{v}|^2 \\ &= |\mathbf{u}|^2 - |\mathbf{v}|^2 \\ &= |\mathbf{u}|^2 - |\mathbf{u}|^2 = 0, \end{aligned}$$

The last line uses the hypothesis $|\mathbf{u}| = |\mathbf{v}|$. This shows that $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal. $\square$



When $|\mathbf{u}| = |\mathbf{v}|$ the parallelogram is a rhombus. The problem shows that the diagonals of the rhombus are orthogonal.