

Optimization

Name: _____

Preview Activity for Friday 10/9 · Submit via Gradescope by 11:59 p.m. Thursday 10/8.

We learn in single-variable calculus that the derivative is a useful tool for finding the local maxima and minima of functions, and that these ideas may often be employed in applied settings. In particular, if a function f , such as the one shown in Figure 1, is everywhere differentiable, we know that the tangent line is horizontal at any point where f has a local maximum or minimum. This, of course, means that the derivative f' is zero at any such point. Hence, one way that we seek extreme values of a given function is to first find where the derivative of the function is zero.

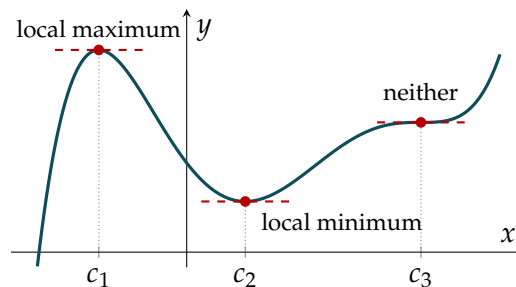


Figure 1. The graph of $y = f(x)$, with horizontal tangent lines at its critical points c_1 , c_2 , and c_3 .

In multivariable calculus, we are often similarly interested in finding the greatest and/or least value(s) that a function may achieve. Moreover, there are many applied settings in which a quantity of interest depends on several different variables. In the following preview activity, we begin to see how some key ideas in multivariable calculus can help us answer such questions by thinking about the geometry of the surface generated by a function of two variables.

Activity _____ Local maxima of functions of two variables¹

Let $z = f(x, y)$ be a differentiable function, and suppose that at the point (x_0, y_0) , f achieves a local maximum. That is, the value of $f(x_0, y_0)$ is greater than the value of $f(x, y)$ for all (x, y) nearby (x_0, y_0) . You might find it helpful to sketch a rough picture of a possible function f that has this property.

- If we consider the trace given by holding $y = y_0$ constant, then the single-variable function defined by $f(x, y_0)$ must have a local maximum at x_0 . What does this say about the value of the partial derivative $f_x(x_0, y_0)$?
- In the same way, the trace given by holding $x = x_0$ constant has a local maximum at $y = y_0$. What does this say about the value of the partial derivative $f_y(x_0, y_0)$?

(c) What may we now conclude about the gradient $\nabla f(x_0, y_0)$ at the local maximum? How is this consistent with the statement “ f increases most rapidly in the direction $\nabla f(x_0, y_0)$ ”?

(d) How will the tangent plane to the surface $z = f(x, y)$ appear at the point $(x_0, y_0, f(x_0, y_0))$?

(e) By first computing the partial derivatives, find any points at which

$$f(x, y) = 2x - x^2 - (y + 2)^2$$

may have a local maximum.

Optional Reading

Stewart §14.7.

¹Preview Activity 10.7.1 of *Active Calculus — Multivariable* (Schlicker, Keller, Long), activecalculus.org, CC BY-SA 4.0.