

Directional Derivatives

Name: _____

Preview Activity for Wednesday 10/7 · Submit via Gradescope before class.

The partial derivatives of a function tell us the instantaneous rate at which the function changes as we hold all but one independent variable constant and allow the remaining independent variable to change. It is natural to wonder how we can measure the rate at which a function changes in directions other than parallel to a coordinate axes. In what follows, we investigate this question, and see how the rate of change in any given direction is connected to the rates of change given by the standard partial derivatives.

Activity _____ Walking through a temperature field¹

Let's consider the function f defined by

$$f(x, y) = 30 - x^2 - \frac{1}{2}y^2,$$

and suppose that f measures the temperature, in degrees Celsius, at a given point in the plane, where x and y are measured in feet. Assume that the positive x -axis points due east, while the positive y -axis points due north. A contour plot of f is shown in Figure 1.

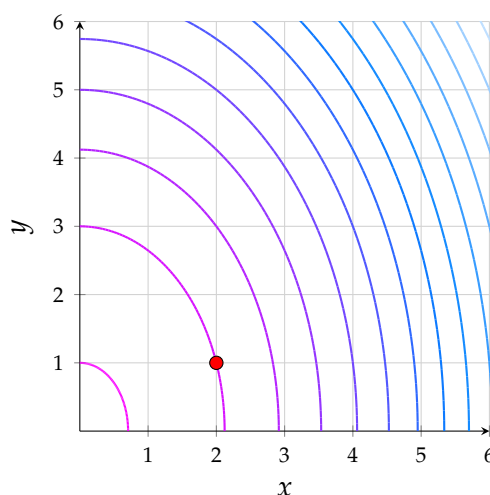


Figure 1. A contour plot of $f(x, y) = 30 - x^2 - \frac{1}{2}y^2$.

- Suppose that a person is walking due east, and thus parallel to the x -axis. At what instantaneous rate is the temperature changing with respect to x at the moment the walker passes the point $(2, 1)$? What are the units on this rate of change?
- Next, determine the instantaneous rate of change of temperature with respect to distance at the point $(2, 1)$ if the person is instead walking due north. Again, include units on your result.
- Now, rather than walking due east or due north, let's suppose that the person is walking with velocity given by the vector $\mathbf{v} = \langle 3, 4 \rangle$, where time is measured in seconds. Note that the person's speed is thus $|\mathbf{v}| = 5$ feet per second. Find parametric equations for the person's path; that is, parameterize the line through $(2, 1)$ using the direction vector $\mathbf{v} = \langle 3, 4 \rangle$. Let $x(t)$ denote the x -coordinate of the line, and $y(t)$ its y -coordinate. Make sure your parameterization places the walker at the point $(2, 1)$ when $t = 0$.

- (d) With the parameterization in (c), we can now view the temperature f as not only a function of x and y , but also of time, t . Hence, use the chain rule to determine the value of $\left. \frac{df}{dt} \right|_{t=0}$. What are the units on your answer? What is the practical meaning of this result?
- (e) Now suppose the person walks through $(2, 1)$ in the same direction as in (c), but at a speed of 1 foot per second, so that the velocity is $\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$. Repeat (c) and (d) with $\mathbf{u}$ in place of $\mathbf{v}$. What are the units on your answer now, and how does it compare to your answer in (d)?
- (f) Your computation in (e) can be written as a dot product $\langle _, _ \rangle \cdot \mathbf{u}$, where the first vector depends only on f and the point $(2, 1)$, and $\mathbf{u}$ depends only on the direction you walk. Which unit vector $\mathbf{u}$ makes this dot product as large as possible? Sketch that direction at $(2, 1)$ on Figure 1. How does it relate to the contour through $(2, 1)$?

Optional Reading

Stewart §14.6.

¹Preview Activity 10.6.1 of *Active Calculus — Multivariable* (Schlicker, Keller, Long), activecalculus.org, CC BY-SA 4.0. Parts (e) and (f) are not part of the original activity.