

Derivatives of Vector Functions

Name: _____

Preview Activity for Friday 9/18 · Submit via Gradescope before class.

A **vector-valued function** is a function whose domain is a set of real numbers and whose outputs are vectors. In $\mathbb{R}^3$, a vector-valued function has the form

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k},$$

where f , g , and h are ordinary real-valued functions which we call the *component functions* of $\mathbf{r}$. A vector function is actually the same data as the parametric equations $x = f(t)$, $y = g(t)$, $z = h(t)$. If we draw each vector $\mathbf{r}(t)$ starting at the origin, its tip is the point $(f(t), g(t), h(t))$; as t varies the tips trace out the parametric curve. The vector equation $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$ of a line through the tip of $\mathbf{r}_0$ in the direction of $\mathbf{v}$ is an example of a vector-valued function. Here is a visualization of a vector-valued function: [GeoGebra: Vector-Valued Function](#). The function is:

$$\mathbf{r}(t) = \langle t^2 \sin(t) \cos(t), 0.9 \cos(t^2), \sin(t) \rangle, \quad 0 \leq t \leq 2\pi.$$

The output $\mathbf{r}(t)$ is the blue arrow, which is the position vector of the point $A = (t^2 \sin(t) \cos(t), 0.9 \cos(t^2), \sin(t))$ on the curve at time t .

Review: Before class, you should review the derivative of a function $f(x)$ (Stewart §2.7). In particular, try to understand why the derivative $f'(x)$ measures the slope of the tangent line to the graph of f at the point $(x, f(x))$. Here is a visualization of what's going on: [GeoGebra: Tangent Line Approximation by Secant Line](#). Notice that the difference quotient $\frac{f(x+h) - f(x)}{h}$ is the slope of the secant line through the points $(x, f(x))$ and $(x+h, f(x+h))$. As we take $h \rightarrow 0$ in the limit, the secant line approaches the tangent line. Thus,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

is the slope of the tangent line to the graph of f at the point $(x, f(x))$. We will be generalizing this idea to define the derivative of a vector-valued function.

Activity _____ Derivatives of vector-valued functions¹

Let $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(2t)\mathbf{j}$ describe the path traveled by an object at time t .

- Use appropriate technology (like [GeoGebra](#)) to help you sketch the graph of the vector-valued function $\mathbf{r}$, and then locate and label the point on the graph when $t = \pi$.
- Recall that for functions of a single variable, the derivative of a sum is the sum of the derivatives; that is, $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$. With this idea in mind and viewing $\mathbf{i}$ and $\mathbf{j}$ as constant vectors, what do you expect the derivative of $\mathbf{r}$ to be? Write a proposed formula for $\mathbf{r}'(t)$.
- Use your result from part (b) to compute $\mathbf{r}'(\pi)$. Sketch this vector $\mathbf{r}'(\pi)$ as emanating from the point on the graph of $\mathbf{r}$ when $t = \pi$, and explain what you think $\mathbf{r}'(\pi)$ tells us about the object's motion.

Optional Reading

Stewart §13.1 and §13.2.

¹Preview Activity 9.7.1 of *Active Calculus — Multivariable* (Schlicker, Keller, Long), activecalculus.org, CC BY-SA 4.0.