

The Cross Product

Name: _____

Preview Activity for Wednesday 9/9 · Submit via Gradescope by 11:59 p.m. the night before class.

So far, we have encountered some basic algebraic operations on vectors, such as addition, subtraction, scalar multiplication, and the dot product, and explored their geometric interpretations. In this section, we will encounter another algebraic operation, the *cross product*, which likewise carries geometric meaning; in fact, both its length and its direction will tell us something about the two vectors we started with. Whereas the operations above make sense in any dimension, we must emphasize that the cross product is only defined for vectors in $\mathbb{R}^3$. Recall that we will always be working with a right-hand system, as described in Stewart §12.1. In particular, the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are oriented as in Figure 1: if we point the index finger of our right hand along $\mathbf{i}$ and our middle finger along $\mathbf{j}$, then our thumb points along $\mathbf{k}$.

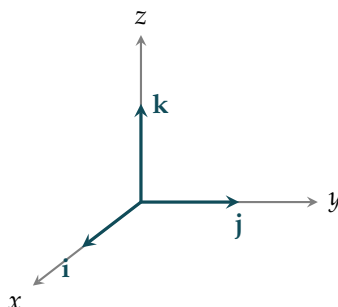


Figure 1. Basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

We saw in class that every vector in $\mathbb{R}^3$ is a sum of scalar multiples of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$; for instance, $\langle 2, 3, -1 \rangle = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$. You can use [this applet](#) to visualize this idea. Our strategy for defining the cross product is to define it first on $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, and then extend it to sums and scalar multiples. You will explore this idea in the following activity.

Activity _____ The cross product¹

The cross product of two vectors, $\mathbf{u}$ and $\mathbf{v}$, will itself be a *vector* denoted $\mathbf{u} \times \mathbf{v}$. The direction of $\mathbf{u} \times \mathbf{v}$ is determined by the right-hand rule: if we point the index finger of our right hand in the direction of $\mathbf{u}$ and our middle finger in the direction of $\mathbf{v}$, then our thumb points in the direction of $\mathbf{u} \times \mathbf{v}$.

- We begin by defining the cross products using the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. Referring to Figure 1, explain why $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in that order form a right-hand system. We then define $\mathbf{i} \times \mathbf{j}$ to be $\mathbf{k}$ — that is $\mathbf{i} \times \mathbf{j} = \mathbf{k}$.
- Now explain why $\mathbf{i}$, $\mathbf{k}$, and $-\mathbf{j}$ in that order form a right-hand system. We then define $\mathbf{i} \times \mathbf{k}$ to be $-\mathbf{j}$ — that is $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

¹Preview Activity 9.4.1 of *Active Calculus — Multivariable* (Schlicker, Keller, Long), activecalculus.org, CC BY-SA 4.0.

- (c) Continuing in this way, complete the missing entries in Table 1.

$\mathbf{i} \times \mathbf{j} = \mathbf{k}$	$\mathbf{i} \times \mathbf{k} = -\mathbf{j}$	$\mathbf{j} \times \mathbf{k} =$
$\mathbf{j} \times \mathbf{i} =$	$\mathbf{k} \times \mathbf{i} =$	$\mathbf{k} \times \mathbf{j} =$

Table 1. Table of cross products involving $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

- (d) Up to this point, the products you have seen, such as the product of real numbers and the dot product of vectors, have been commutative, meaning that the product does not depend on the order of the terms. For instance, $2 \cdot 5 = 5 \cdot 2$. The table above suggests, however, that the cross product is *anti-commutative*: for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$. If we consider the case when $\mathbf{u} = \mathbf{v}$, this shows that $\mathbf{v} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{v})$. What does this tell us about $\mathbf{v} \times \mathbf{v}$; in particular, what vector is unchanged by scalar multiplication by -1 ?

- (e) It is not difficult to show that the cross product interacts with scalar multiplication and vector addition as one would expect: that is

$$\begin{aligned} (c\mathbf{u}) \times \mathbf{v} &= c(\mathbf{u} \times \mathbf{v}) \\ (\mathbf{u} + \mathbf{v}) \times \mathbf{w} &= (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w}) \end{aligned}$$

We can combine these properties to make cross product calculations a bit easier. For example,

$$\begin{aligned} (2\mathbf{i} + \mathbf{j}) \times \mathbf{k} &= (2\mathbf{i} \times \mathbf{k}) + (\mathbf{j} \times \mathbf{k}) \\ &= 2(\mathbf{i} \times \mathbf{k}) + (\mathbf{j} \times \mathbf{k}) \\ &= -2\mathbf{j} + \mathbf{i}. \end{aligned}$$

Using these properties along with Table 1, find the cross product $\mathbf{u} \times \mathbf{v}$ if $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{v} = -\mathbf{i} + \mathbf{k}$.

- (f) Verify that the cross product $\mathbf{u} \times \mathbf{v}$ you just found in part (e) is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

- (g) Consider the vectors $\mathbf{u}$ and $\mathbf{v}$ in the xy -plane as shown below in Figure 2.

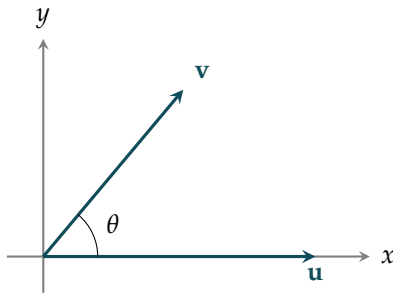


Figure 2. Two vectors in the xy -plane.

Explain why $\mathbf{u} = |\mathbf{u}|\mathbf{i}$ and $\mathbf{v} = |\mathbf{v}|\cos(\theta)\mathbf{i} + |\mathbf{v}|\sin(\theta)\mathbf{j}$. Then compute the length $|\mathbf{u} \times \mathbf{v}|$.

Optional Reading

Stewart §12.4.