The Dot Product

Name: ____________________

Preview Activity for Friday 9/4 · Submit via Gradescope by 11:59 p.m. the night before class.

Consider the two-dimensional vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$. The *dot product* of $\mathbf{u}$ and $\mathbf{v}$ is denoted $\mathbf{u} \cdot \mathbf{v}$ and defined to be

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

For instance, we have $\langle 2, -1 \rangle \cdot \langle 1, 4 \rangle = 2 \cdot 1 + (-1) \cdot 4 = -2$. At first glance, this may feel like we've *lost* information, as we have taken two objects with both magnitude and direction, multiplied them, and obtained a scalar. However, this scalar allows us to draw many useful conclusions, as you will now begin exploring.

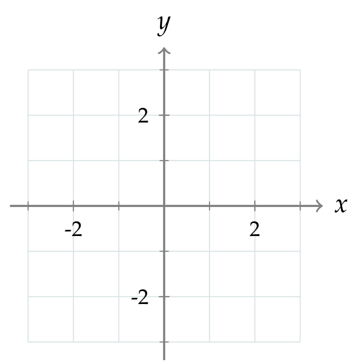
You may find this GeoGebra applet useful as you work through the activity; it lets you drag two vectors around and watch the dot product change.

Activity — The dot product and the angle between vectors[1]

In this activity, we will use the following vectors:

$$\mathbf{u}_1 = \langle 1, 2 \rangle, \quad \mathbf{u}_2 = \langle -1, 1 \rangle, \quad \mathbf{u}_3 = \langle 2, -1 \rangle, \quad \mathbf{u}_4 = \langle 1, -2 \rangle$$

(a) Draw representatives of each of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ in standard position using the axes below.



(b) Compute each of the dot products listed in the table below. Fill in only the "Value" column at this point. The "Angle Classification" column will be completed in the next part.

Dot Product	Value	Angle Classification
$\mathbf{u}_1 \cdot \mathbf{u}_2$		
$\mathbf{u}_1 \cdot \mathbf{u}_3$		
$\mathbf{u}_1 \cdot \mathbf{u}_4$		
$\mathbf{u}_2 \cdot \mathbf{u}_4$		
$\mathbf{u}_3 \cdot \mathbf{u}_4$		

[1] This is Preview Activity 9.4.1 of *Active Calculus — Multivariable*, 2nd ed., by Nicholas Long and Mitchel T. Keller, activecalculus.org, used under CC BY-SA 4.0.

(c) When we look at vectors drawn with the same initial point, as you have done with the vectors above, we can consider the angle between two vectors to be the *smaller* angle formed by looking at them. This results in an angle between 0 and π if we measure in radians or between 0° and 180° if we measure in degrees. Rather than measuring the angles between the vectors you've drawn precisely, classify the angle between the pairs of vectors in the table in part (b) by writing acute, right, or obtuse in the "Angle Classification" column.

(d) Look at the values of the dot products that you computed in part (b) and the classifications of the angles between the vectors that you made in part (c). Write a sentence or two to describe what you notice about the relationship between the *sign* of the dot product and the type of angle between the vectors.

(e) Does the value of the dot product or the classification of the angle change if we change the order in which we consider the vectors?

(f) Do you think that knowing only the value of the dot product $\mathbf{v} \cdot \mathbf{w}$ would be enough to determine the exact value of the angle between $\mathbf{v}$ and $\mathbf{w}$ if you didn't know the components of $\mathbf{v}$ and $\mathbf{w}$? Write a sentence to explain your reasoning.

Hint. Compare the pair of vectors $\mathbf{u}_1$ and $\mathbf{u}_4$ to $\mathbf{u}_3$ and $\mathbf{u}_4$.

(g) Although the angle between a vector and itself is 0, the dot product of a vector with itself can be computed. Compute $\mathbf{u}_1 \cdot \mathbf{u}_1$ and $\mathbf{u}_2 \cdot \mathbf{u}_2$ as well as the magnitude of $\mathbf{u}_1$ and $\mathbf{u}_2$. What do you notice about the relationship between these values? Write a sentence explaining why this seems reasonable based on the way the dot product of a vector with itself and the magnitude of that vector are computed.