

Midterm Exam 2 Info

Topics: The midterm exam on Friday, October 16 will cover Sections 14.1–14.8 of Stewart. Differentials (a topic from 14.4) will not be on the exam as we never discussed them. Anything else covered in the textbook sections, the preview activities, the worksheets, or discussed in class is fair game. However, topics explicitly discussed in class will be prioritized.

What to expect: You will have the entire class period to work on the exam. The format will be similar to Midterm 1. Expect a mix of question types: computational problems in the style of the quizzes, short questions worth only a point or two, true/false questions requiring no justification, questions that ask you to work with a picture (reading a contour map, sketching a trace, drawing a gradient vector), and questions that ask you to explain your reasoning (for instance, why the gradient is perpendicular to a level curve, or why a critical point must be checked rather than assumed to be an extremum). The worksheets are designed for learning new ideas and WebAssign is designed for practice with instant feedback, while the exam is designed to measure the learning objectives listed below. For this reason, the exam questions may look different from both. Keep in mind that no notes or calculators are allowed on the exam, so you will need to be able to recall some formulas.

If you are short on time: consider prioritizing the topics in this order: partial derivatives (14.3), the gradient and directional derivatives (14.6), optimization (14.7), Lagrange multipliers (14.8). After those, linearization and tangent planes (14.4), functions of several variables (14.1), the chain rule (14.5), and limits (14.2). However, any topic is fair game.

Suggestion for studying: Reflect on each item below. Can you do it right now, without looking at your notes? Test yourself. Find a problem of that type and try to solve it: the exercises at the end of each section of the textbook are organized by topic, and the odd-numbered ones have the answers in the back. Try the quiz problems again without looking at the solutions, and try the worksheet problems again without any of the scaffolding (they are usually broken down into steps for you). Worksheet 18 is a page of optimization problems with no scaffolding at all; if you can do those six, you are in good shape for the last two sections.

Here is my (non-exhaustive) summary of the specific things you should know and be able to do from each section of the book.

1. Functions of several variables (14.1):

- (a) Know what a function of two (or more) variables is. Find and sketch the domain of such a function, and find its range.
- (b) Evaluate a function of two variables, including from a table of values.
- (c) Know what the graph of $f(x, y)$ is, and that it is a surface in $\mathbb{R}^3$.
- (d) Find the traces $z = f(x, c)$ and $z = f(c, y)$ of a function, and sketch them in the xz - and yz -planes.
- (e) Know the definition of a level curve and a contour map. Sketch the level curves $f(x, y) = k$ for several values of k , and label them.
- (f) Use traces and level curves together to sketch or identify the graph of a function, and match a surface to its contour map.
- (g) Read a contour map: estimate values, decide which direction is uphill, compare steepness in different directions, and explain what the spacing of the level curves says about the graph.

2. Limits and continuity (14.2):

- (a) Have basic intuition about the limit of a multivariable function. You do not need to know the precise definition.

- (b) Compute a limit when it exists, using properties of limits and continuity.
- (c) Compute a limit when it exists by simplifying algebraically, by squeezing, or by changing variables.
- (d) Show that a limit does not exist using “limits along different paths”. Parametrize a path (an axis, a line $y = mx$, a parabola $y = x^2$ or $x = y^2$) and compute the limit along it.
- (e) Show that a limit does not exist by reading a contour diagram.
- (f) Know that paths can only disprove a limit, never prove one: agreement along every line is not enough.
- (g) Determine whether a function is continuous at a point or on a specified set. Find points of discontinuity of a function.

3. **Partials (14.3):**

- (a) Know the limit definition of a partial derivative, and the notations $\partial f / \partial x$ and f_x .
- (b) Know what a partial derivative measures geometrically.
- (c) Compute partial derivatives using differentiation rules from single-variable calculus, including for functions of three or more variables.
- (d) Interpret partial derivatives in specific contexts (e.g. units and physical interpretations).
- (e) Estimate partial derivatives using a table of data or the level curves of a function, using the smallest available step.
- (f) Determine whether a partial derivative is positive, negative, or zero at a point using a contour diagram.
- (g) Compute second-order (or higher-order) partial derivatives using differentiation rules from single-variable calculus. Know Clairaut’s Theorem ($f_{xy} = f_{yx}$ when both are continuous).
- (h) Use Clairaut’s Theorem to choose the easier order in which to differentiate when computing a mixed partial.
- (i) Understand what f_{xx} and f_{yy} measure geometrically (the concavity of the traces).

4. **Linearization (14.4):**

- (a) Determine if a function is locally linear (the tangent plane exists and approximates the function) using the notion of continuous differentiability. Know that having both partial derivatives at a point is not enough.
- (b) Compute an equation of a tangent plane to the graph of a differentiable function, and know where the formula comes from (the two trace tangent vectors and their cross product).
- (c) Use the tangent plane as a linear approximation of a function: $L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$.
- (d) Find a linear approximation of a function using a table of values or a contour diagram. This requires that you estimate the first-order partial derivatives. Estimate the value of the function using the linear approximation.

5. **Chain rule (14.5):**

- (a) Construct a tree diagram and use the tree diagram to write down a formula for the chain rule.
- (b) Use the chain rule to compute derivatives and partial derivatives of compositions of functions, and check your answer by substituting when that is reasonable.
- (c) Use implicit differentiation, $dy/dx = -F_x/F_y$, to differentiate a relation such as $x^3 + y^3 = 6xy$.

6. **Directional derivative (14.6):**

- (a) Compute the gradient of a multivariable function, for two, three, or more variables.
- (b) Compute the directional derivative using the gradient and the dot product, including when

the direction is given as a non-unit vector or as two points. Know why the direction vector must be a unit vector.

- (c) Understand what the directional derivative measures geometrically. Know that f_x and f_y are the directional derivatives in the directions $\mathbf{i}$ and $\mathbf{j}$.
- (d) Determine the instantaneous rate of change of a function in a certain direction.
- (e) Determine the direction of steepest ascent or descent using the gradient, as a unit vector, and determine the rate of change in that direction.
- (f) Find a direction in which the derivative of f is zero.
- (g) Know and be able to explain why ∇f is perpendicular to the level curve through a point. Draw gradient vectors on a contour diagram, and use the picture to say whether f is increasing or decreasing in a given direction.
- (h) Compute the equation of a tangent plane to a graph of a function using the gradient, and find the tangent plane to a level surface.

7. Optimization (14.7):

- (a) Find critical points of a function of two variables: $\nabla f = \mathbf{0}$, or a partial derivative fails to exist. Find *all* of them.
- (b) Know what the various types of extrema (local max/min, global max/min) are and what they look like geometrically. Know what a saddle point is.
- (c) Know that local maxima and minima occur only at critical points, and that a critical point need not be an extremum.
- (d) Classify critical points using the Second-Derivative Test, computing the Hessian and the discriminant $D = f_{xx}f_{yy} - f_{xy}^2$. Know that the test gives no information when $D = 0$, and be able to give an example where that happens.
- (e) Know the definitions of a closed region and a bounded region, and decide whether a given region is each.
- (f) Know what the Extreme Value Theorem says, and know why both hypotheses matter.
- (g) Optimize a function on a closed and bounded region. This involves finding critical points on the interior of the region and on the boundary, as well as the corners or endpoints, then comparing the values. On the boundary you can use Lagrange multipliers, or you can parametrize and do Calculus I.

8. Lagrange multipliers (14.8):

- (a) Explain geometrically why the extreme values occur where a contour of f is tangent to the constraint curve.
- (b) Know Lagrange's Theorem, the method of Lagrange multipliers, and what a constrained critical point is.
- (c) Set up and solve the system $\nabla f = \lambda \nabla g$, $g = k$, for two or more variables, finding *all* solutions.
- (d) Use Lagrange multipliers for the boundary step of a closed and bounded problem, remembering that the interior still needs $\nabla f = \mathbf{0}$.
- (e) Set up a word problem (a box, a can, a closest point) as a constrained optimization problem, deciding which function is f and which is g .
- (f) Decide which constrained critical point is the maximum and which is the minimum. When the constraint set is closed and bounded, cite the Extreme Value Theorem; when it is not, argue from the behavior of f along the constraint.