Brauer pairs for chain complexes

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Representations of Finite Groups and Related Structures
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In honor of Robert Boltje's 65th birthday

Motivation

$F =$ field of char. $p > 0$; G, H finite; A, B blocks of FG, FH.

Conjecture (Broué 1990, Rickard 1996)

If A has abelian defect group D, then A is splendidly derived equivalent to its Brauer correspondent.

Splendid: the terms of the Rickard complex are p-permutation bimodules.

Problem (R. Boltje)

Determine the structure and invariants of a splendid Rickard equivalence.

A splendid Rickard equivalence C induces a p-permutation equivalence:

$$C \in K^b({}_A\mathbf{triv}_B) \xrightarrow{\gamma_C = \sum_n (-1)^n [C_n]} \gamma_C \in T^\Delta(A, B).$$

- **Boltje–Perepelitsky '25:** studied p-permutation equivalences via a poset of Brauer pairs $\mathcal{BP}(\gamma)$.
- **Breland–Miller '26:** defined $\mathcal{BP}(C)$ for a splendid Rickard C; proved $\mathcal{BP}(C) = \mathcal{BP}(\gamma_C)$.

This talk

Study $\mathcal{BP}(C)$ for an arbitrary complex C of p-permutation modules. Specialize to splendid Rickard.

Background

Brauer pairs

$$\mathcal{BP}(FG) = \{(P,e) : P \in \mathcal{S}_p(G),\ e \in \mathrm{Bl}(FC_G(P))\}$$
$$= \coprod_{A \in \mathrm{Bl}(FG)} \mathcal{BP}(A), \text{ a } G\text{-poset under } \leq.$$

Brauer pairs of a p-permutation module

For $M \in {}_{FG}\mathbf{triv}$ indecomposable, $\mathcal{BP}(M) = \{(P,e) : M(P,e) := eM(P) \neq 0\}$.

- $\mathcal{BP}(M)$ is a G-stable ideal in $\mathcal{BP}(FG)$;
- (P,e) maximal $\iff$ P a vertex of M (all maximal pairs G-conjugate).

Brauer pairs for chain complexes

Brauer pairs for chain complexes

Let C be a bounded complex of p-permutation FG-modules, $(P, e) \in \mathcal{BP}(FG)$.

- $(P, e) \in \mathcal{BP}(C) \iff C(P, e)$ not contractible (C-Brauer pair);
- $(P, e) \in \mathcal{BP}^c(C) \iff \mathrm{Res}^{N_G(P,e)}_{C_G(P)} C(P, e)$ not contractible (centralizer C-Brauer pair);
- $(P, e) \in \mathcal{BP}^a(C) \iff C(P, e)$ not acyclic (non-acyclic C-Brauer pair).

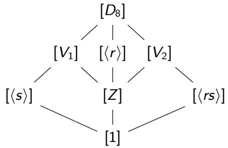
- Generalizes $\mathcal{BP}(M)$; for a module all three coincide and form an ideal.
- In general, $\mathcal{BP}^a(C) \subseteq \mathcal{BP}^c(C) \subseteq \mathcal{BP}(C)$, all G-stable.

Question

Does the poset of Brauer pairs for a p-permutation complex behave similarly to the poset of Brauer pairs for a p-permutation module?

Example: $G = D_8 = \langle r, s\rangle$, $p = 2$, $V_1 = \langle r^2, s\rangle$, $Z = \langle r^2\rangle$

$$C = 0 \to F \xrightarrow{\binom{1}{1}} F[G/V_1] \xrightarrow{(1\ 1)} F \to 0, \quad \text{acyclic, not contractible.}$$



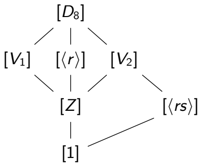
Fact 1: If $P \not\leq_G V_1$, then $C(P) = 0 \to F \to 0 \to F \to 0$ (not acyclic).
Fact 2: If $P \leq_G V_1$, then $C(P) \cong \mathrm{Res}^G_{N_G(P)} C$

$\mathcal{BP}^c(C)$: $P \leq_G V_1$, use $C_G(P)$:

- $C_G(1) = C_G(Z) = G$: $C \not\simeq 0$, so $1, Z \in \mathcal{BP}^c(C)$;
- $C_G(\langle s\rangle) = C_G(V_1) = V_1$: $\mathrm{Res}_{V_1} C \simeq 0$, so $\langle s\rangle, V_1 \notin \mathcal{BP}^c(C)$.

$\mathcal{BP}(C)$: recheck $V_1, \langle s\rangle$ via $N_G(P)$:

- $N_G(V_1) = G$: $C(V_1) \cong C \not\simeq 0$, so $V_1 \in \mathcal{BP}(C)$;
- $N_G(\langle s\rangle) = V_1$: $C(\langle s\rangle) \cong \mathrm{Res}_{V_1} C \simeq 0$, so $\langle s\rangle \notin \mathcal{BP}(C)$.



$\mathcal{BP}^a(C)/G\mathcal{BP}^c(C)/G\mathcal{BP}(C)/G$

Example: continued

$$
\begin{tikzcd}[column sep=small]
{[D_8]} \arrow[d, dash] \arrow[dr, dash] & & \\
{[\langle r\rangle]} & {[V_2]} \arrow[dr, dash] & \\
& & {[\langle rs\rangle]}
\end{tikzcd}
$$

$\mathcal{BP}^a(C)/G$

$$
\begin{tikzcd}[column sep=small]
{[D_8]} \arrow[d, dash] \arrow[dr, dash] & & \\
{[\langle r\rangle]} \arrow[d, dash] & {[V_2]} \arrow[dl, dash] \arrow[dr, dash] & \\
{[Z]} \arrow[d, dash] & & {[\langle rs\rangle]} \arrow[dll, dash] \\
{[1]} & &
\end{tikzcd}
$$

$\mathcal{BP}^c(C)/G$

$$
\begin{tikzcd}[column sep=small]
& {[D_8]} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & & \\
{[V_1]} \arrow[dr, dash] & {[\langle r\rangle]} \arrow[d, dash] & {[V_2]} \arrow[dl, dash] \arrow[dr, dash] & \\
& {[Z]} \arrow[d, dash] & & {[\langle rs\rangle]} \arrow[dll, dash] \\
& {[1]} & &
\end{tikzcd}
$$

$\mathcal{BP}(C)/G$

- Strict inclusions $\mathcal{BP}^a(C) \subsetneq \mathcal{BP}^c(C) \subsetneq \mathcal{BP}(C)$.
- None of $\mathcal{BP}^a(C)$, $\mathcal{BP}^c(C)$, $\mathcal{BP}(C)$ are ideals in $\mathcal{BP}(FG)$.

Questions:

When is $\mathcal{BP}^-(C)$ a G-stable ideal? When do we have equality?

When is the poset an ideal?

Theorem (B.)

Let C be a p-permutation complex. Then

$$\mathcal{BP}^c(C) \text{ is a } G\text{-stable ideal} \iff \mathcal{BP}^c(C) = \mathcal{BP}(C)$$

Examples.

- modules and 2-term complexes (there $\mathcal{BP}^a(C) = \mathcal{BP}(C)$);
- C belongs to a block with abelian defect group — the setting of Broué's conjecture (here $\mathcal{BP}^c(C) = \mathcal{BP}(C)$).

Why do we care?

When the ideal property holds, $\mathcal{BP}^{(-)}(C)$ controls the vertices of the constituents of C; if its maximal elements are conjugate, a single maximal pair does.

Main theorem: ideal property

Recall: splendid Rickard equivalence

Bounded complex C of p-permutation (A, B)-bimodules with $C \otimes_B C^* \simeq A$, $C^* \otimes_A C \simeq B$ in K^b, whose terms have twisted-diagonal vertices $\Delta(P, \psi, Q)$.

Theorem (B.–Miller)

Let C be a splendid Rickard equivalence for A, B and $\lambda = (\Delta(P, \psi, Q), e \otimes f^*) \in \mathcal{BP}^{\Delta}(A, B)$. TFAE:

(a) $\lambda \in \mathcal{BP}^c(C)$;

(b) $eC(\Delta(P, \psi, Q))f$ *is a splendid Rickard equivalence between* $F[C_G(P)]e$ *and* $F[C_H(Q)]f$;

(c) $\lambda \in \mathcal{BP}(C)$;

(d) $eC(\Delta(P, \psi, Q))f \mathbin{\widetilde{\otimes}} (eC(\Delta(P, \psi, Q))f)^* \simeq F[C_G(P)]e$ *(extended tensor product).*

$(a) \Rightarrow (b)$: apply $e\,(-)(\Delta(P))\,e$ to $A \simeq C \otimes_B C^*$:

$$F[C_G(P)]e \;\cong\; e\,A(\Delta(P))\,e \;\simeq\; e\,(C \otimes_B C^*)(\Delta(P))\,e$$

$$\overset{\text{Boltje–Perepelitsky}}{\cong} \bigoplus_{(\psi,(Q,f))} eC(\Delta(P, \psi, Q))f \otimes_{F[C_H(Q)]} \big(eC(\Delta(P, \psi, Q))f\big)^*$$

(Krull–Schmidt) $F[C_G(P)]e$ indecomposable $\Rightarrow$ a unique orbit survives, so $eC(\Delta(P, \psi, Q))f$ is a splendid Rickard equivalence between the local blocks. $\Rightarrow$ (b).

Properties and structural control

Let A, B be blocks, C a splendid Rickard equivalence, and write $\lambda = (\Delta(P, \psi, Q), e \otimes f^*)$.

Theorem (B.–Miller)

1. **(Sylow)** *$G \times H$ acts transitively on the maximal elements of $\mathcal{BP}(C)$.*
2. **(Defect & fusion)** *λ maximal $\iff$ P is a defect group of A and Q is a defect group of B. In that case, $\psi\colon Q \xrightarrow{\sim} P$ is an isomorphism of block fusion systems $\mathcal{F}_{(Q,f)}(B) \xrightarrow{\sim} \mathcal{F}_{(P,e)}(A)$.*
3. **(Vertex control)** *If λ is maximal, then every constituent of C has a maximal Brauer pair contained in λ; constituent vertices are diagonal w.r.t. ψ.*

Upshot. The techniques used carry over from splendid Rickard to splendid *stable* (and relatively stable) equivalences, giving a local-to-global characterization of such C.

The End.

Thank you.

A local-to-global theorem

A splendid stable equivalence is a splendid complex C with

$$C \otimes_B C^* \cong A \oplus Z, \qquad C^* \otimes_A C \cong B \oplus Z',$$

where Z, Z' are homotopy equivalent to complexes of projectives.

Theorem (B.)

Let C be a splendid complex of p-permutation (A, B)-bimodules. Then C is a splendid stable equivalence if and only if

- **(Ideal)** *the nontrivial Brauer pairs $\mathcal{BP}(C)_{>1}$ form a $G \times H$-stable ideal in $\mathcal{BP}^{\Delta}_{>1}(A, B)$;*
- **(Sylow)** *$G \times H$ acts transitively on the maximal elements of $\mathcal{BP}(C)_{>1}$;*
- **(Fusion)** *for a maximal pair $(\Delta(D, \varphi, E), e_D \otimes f_E^*)$, $\varphi\colon E \xrightarrow{\sim} D$ is an isomorphism of block fusion systems $\mathcal{F}_{(E,f_E)}(B) \xrightarrow{\sim} \mathcal{F}_{(D,e_D)}(A)$;*
- **(Local equivalences)** *for every $(\Delta(P, \psi, Q), e \otimes f^*) \in \mathcal{BP}(C)_{>1}$, the Brauer construction $eC(\Delta(P, \psi, Q))f$ is a splendid Rickard equivalence between $F[C_G(P)]e$ and $F[C_H(Q)]f$;*

This is a special case of a more general local-to-global theorem for splendid relatively stable equivalences.